



# Toward a General Rewriting-Based Framework for Reducibility

Colin Riba

## ► To cite this version:

| Colin Riba. Toward a General Rewriting-Based Framework for Reducibility. 2008. hal-00779623

**HAL Id: hal-00779623**

**<https://hal.science/hal-00779623>**

Submitted on 22 Jan 2013

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Toward a General Rewriting-Based Framework for Reducibility

Colin Riba

INRIA Sophia Antipolis – Méditerranée \*

`Colin.Riba@sophia.inria.fr`

December 18, 2008

Reducibility is a powerful proof method which applies to various properties of typed terms in different type systems. For strong normalization, different variants are known, such as Girard’s reducibility candidates, Tait’s saturated sets and biorthogonals. They differ by the closure conditions imposed to types interpretations, called here reducibility families.

This paper is about the computational and observational properties underlying untyped reducibility. Our starting point is the comparison of reducibility families w.r.t. their ability to handle rewriting, for which their possible stability by union plays an important role. Indeed, usual saturated sets are generally stable by union, but with rewriting it can be difficult to define a uniform notion of saturated sets. On the other hand, rewriting is more naturally taken into account by reducibility candidates, but they are not always stable by union. It seems that for a given rewrite relation, the stability by union of reducibility candidates should imply the ability to naturally define corresponding saturated sets. In this paper, we seek to devise a general framework in which the above claim can be substantiated. In particular, this framework should be as simple as possible, while allowing the formulation of general notions of reducibility candidates and saturated sets.

We present a notion of non-interaction which allows to define neutral terms and reducibility candidates in a generic way. This notion can be formulated in a very simple and general framework, based only on a rewrite relation and a set of contexts, called elimination contexts, required to satisfy some simple properties. This provides a convenient level of abstraction to prove fundamental properties of reducibility candidates, to compare them with biorthogonals, and to study their stability by union. Moreover, we propose a general form of saturated sets, issued from the stability by union of reducibility candidates.

---

\*INRIA Sophia Antipolis - Méditerranée, Everest project, 2004 route des Lucioles - BP 93, 06902 Sophia Antipolis Cedex, France

# Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
<b>2</b>	<b>Preliminaries</b>	<b>4</b>
2.1	Terms, Reductions and Typing . . . . .	4
2.2	Reducibility . . . . .	7
<b>3</b>	<b>Toward Saturated Sets</b>	<b>9</b>
3.1	Basic Mechanisms with Products . . . . .	9
3.2	Pure Lambda-Calculus . . . . .	11
3.3	Lambda-Calculus with Products . . . . .	14
3.4	The Closure Operator of Saturated Sets . . . . .	16
3.5	Lambda-Calculus with Rewriting . . . . .	17
<b>4</b>	<b>Neutral Terms and Reducibility Candidates</b>	<b>20</b>
4.1	Neutral Terms . . . . .	21
4.2	Reducibility Candidates . . . . .	25
4.3	The Closure Operator of Reducibility Candidates . . . . .	29
4.4	Hereditary Neutral Terms . . . . .	30
4.5	The Structure of Reducibility Candidates . . . . .	32
<b>5</b>	<b>Biorthogonals</b>	<b>36</b>
5.1	Orthogonality as a Galois Connection . . . . .	36
5.2	Biorthogonals for Reducibility . . . . .	38
<b>6</b>	<b>Stability by Union</b>	<b>39</b>
6.1	Closure by Union of a Closure Operator . . . . .	40
6.2	Saturated Sets . . . . .	40
6.3	Reducibility Candidates . . . . .	41
6.4	Closure by Union of Biorthogonals . . . . .	46
<b>7</b>	<b>Back to Reducibility Candidates and Saturated Sets</b>	<b>51</b>
7.1	Reducibility Candidates . . . . .	51
7.2	A Comparison of Reducibility Candidates and Saturated Sets . . . . .	52
7.3	Toward a General Notion of Saturated Sets . . . . .	53
<b>8</b>	<b>Conclusion and Directions for Future Work</b>	<b>55</b>

# 1 Introduction

The most flexible termination proof methods for various extensions of typed  $\lambda$ -calculi use type interpretations. All these methods follow the same pattern, which is a variation of realizability called reducibility and due to Girard [Gir72]. Besides termination, reducibility can also be used to build realizability models of programming languages [Pit00, VM04, Vou04], or to study program extraction from proofs [Kri04].

The reducibility method proceeds in two stages. First, types are interpreted by sets of strongly normalizing terms. Second, the adequacy of the interpretation has to be shown: typable terms must belong to the interpretation of their types. However, not every interpretation is adequate. The adequate ones rely on particular closure conditions, which are called reducibility families in this paper. There are essentially three different kinds of reducibility families: Girard’s reducibility candidates [Gir72], Tait’s saturated sets [Tai75] and interpretations based on biorthogonality [Par97] issued from linear logic [Gir87].

We are interested in the comparison and the understanding of reducibility techniques. General facts are known on the comparison of Girard’s and Tait’s closure conditions [Gal89, Luo90, Wer94]. Moreover, [Gal89] provides a precise study of typed and untyped reducibility. Reducibility proofs have been explored abstractly in [Gal95], with aim of isolating ingredients specific to strong normalization from those which can be generalized to abstract realizability models. Different work, but in a similar framework, seek to obtain strong normalization arguments from abstract realizability models [HO93].

In this paper, we focus on computational and observational properties underlying untyped reducibility. Our approach is to compare reducibility families w.r.t. their stability by union and their ability to handle rewriting. While we use stability by union as a tool to explore reducibility, this property is necessary or desired in some cases [Abe06a, BR06, Tat07].

Let us briefly explain why stability by union is interesting to compare reducibility candidates and saturated sets. Adequacy requires reducibility families to be closed by forms of expansions which preserve some properties such as strong normalization or ”good interaction” with elimination contexts. Roughly speaking, the essential difference between Tait’s and Girard’s conditions is the following. Given an elimination term  $t$  and a closed set  $S$ , for  $t$  to belong to  $S$ , Tait’s conditions require only that the weak-head reduct of  $t$  belongs to  $S$ , while Girard’s conditions require that all reduct of  $t$  belong to  $S$ . The correctness of Tait’s conditions is usually easy to establish for orthogonal reduction systems, and so is their stability by union. On the other hand, Girard’s conditions are always correct but not always stable by union. As a matter of a fact, with rewriting it is in general difficult to define a direct notion of saturated sets, the reason being that some rewrite system do not admit stable by union reducibility families [Rib07c]. Intuitively, the stability by union of Girard’s sets should imply the correctness of the corresponding Tait’s conditions. The goal of this paper is to provide a general framework in which this claim can be substantiated. The difficulty is to devise a framework as simple as possible, in which it makes sense to define general notions of reducibility candidates and saturated sets.

After having presented our notations and given basic definitions in Sect. 2, our first step, carried out in Sect. 3, is to isolate and analyze some elementary computational properties on which saturated sets rely, and to precisely see when and how they are used in proofs of basic and well-known results on saturated sets.

This leads in Sect. 4 to the main contribution of this paper: a generic definition of Girard’s reducibility candidates relying on a *non-interaction property* between some terms and contexts. Our framework only assumes a rewrite relation and a set of contexts, called *elimination contexts*, which satisfy some simple properties. We define a general notion of *neutral terms*, as being the

terms which interact with no elimination contexts. Terms which are not neutral are *observable* since they interact with some elimination contexts. We call them *values*. The notion of non-interaction allows to isolate and prove fundamental properties of reducibility candidates. These ideas have been sketched in [Rib08]. The prominent role of elimination contexts in reducibility has already been underlined [Abe04, Mat05], as well as the importance of their interactions with terms [Kri04]. Moreover, the idea of non-interaction is implicitly present in [Gal95, LS05]. However, this notion seems not to have been previously identified for itself and systematically developed as in this paper. We show in Sect. 5 that it also provides a convenient level of abstraction to define biorthogonals and to sketch a comparison with reducibility candidates.

We apply this framework to the study of stability by union in Sect. 6, and thus generalize results of [Rib07b]. Technically, stability by union relies on standardization. The important role of this property in normalization is well-known, in particular with the study of needed redexes and perpetual reductions (see for instance [vRS95, KOvO01]). One contribution of our approach is that stability by union gives an abstract and order-theoretic point of view on standardization in reducibility, in particular thanks to the principal reduct and the strong principal reduct properties.

Our approach is guided by order-theoretical ideas, such as the notion of specialization pre-order, which leads to elegant characterization of stability by union. These order-theoretic ideas seem new in the study of reducibility for normalization, with the exception of [Gal95], where they are not developed for themselves but used as a basis of a more complex framework using the notions of covers and sheaves. Moreover, we have been inspired by the topological notions used to build realizability models of programming languages in [VM04, Vou04].

We come back in Sect. 7 to the comparison of reducibility candidates and saturated sets. We show that our framework allows to define a form of general saturated sets. They apply exactly when reducibility candidates are stable by union, and have a strong correspondence with them, generalizing that of [Rib07b]. However, they do not relate well with usual saturated sets. To get a precise correspondence, it is possible that a notion such as external redexes [KOvO01] may have been better than the strong principal reduct property. We did not follow this line because the formulation of such a notion needs precise syntactical knowledge on rewrite relations, in particular it is crucial to know how redexes are nested in each other [KOvO01, Mel05].

Finally, we conclude in Sect. 8, where we also present directions for future research.

The paper is based on parts of the PhD thesis of the author [Rib07a] (in French). It extends and presents in a uniform way results and ideas which have been sketched or presented in [Rib07b, Rib07c, Rib08]. We assume familiarity with typed  $\lambda$ -calculus [Bar92], reducibility [Gal89, Kri90] and rewriting [Ter03]. Most of the other notions are presented in full details.

## 2 Preliminaries

This section presents the basic ingredients of the paper. We begin by our notations on (typed)  $\lambda$ -calculus and rewriting in Sect. 2.1. We then define type interpretations and reducibility families in Sect. 2.2. The latter depend on the notion of closure operator, which is also briefly recalled.

Given a set  $A$ ,  $\vec{a}$  denotes a finite sequence of elements of  $A$  of length  $|\vec{a}|$ .

### 2.1 Terms, Reductions and Typing

**Terms.** A *signature*  $\Sigma$  is a family of countable sets  $(\Sigma_n)_{n \in \mathbb{N}}$  such that  $\Sigma_n$  contains algebraic symbols of arity  $n$ . We consider  $\lambda$ -terms with uncurried symbols  $f$  in a signature  $\Sigma$  and variables

in a set  $\mathcal{X} = \{x, y, z, \dots\}$ . They are given by the following grammar:

$$t, u \in \Lambda(\Sigma) ::= x \mid \lambda x. t \mid t u \mid f(t_1, \dots, t_n) \quad \text{where} \quad f \in \Sigma_n.$$

As usual, terms are identified modulo renaming of bound variables ( $\alpha$ -conversion). Let  $\Lambda$  be the set of *pure*  $\lambda$ -terms  $\Lambda(\emptyset)$ . A *substitution* is a function  $\sigma$  from  $\mathcal{X}$  to  $\Lambda(\Sigma)$  of finite domain. The capture avoiding application of the substitution  $\sigma$  to the term  $t$  is written  $t\sigma$  or  $t[\sigma(x_1)/x_1, \dots, \sigma(x_n)/x_n]$  if  $\text{Dom}(\sigma) = \{x_1, \dots, x_n\}$ .

Algebraic symbols can contain the binary symbol  $\langle -, - \rangle$  of pairing and the two unary projections  $\pi_1$  and  $\pi_2$ . This gives the  $\lambda$ -calculus with products. In this paper, it is convenient to describe it by extending the syntax of  $\lambda$ -terms with

$$t, u \in \Lambda(\Sigma) ::= \dots \mid \langle t, u \rangle \mid \pi_1 t \mid \pi_2 t,$$

where it is assumed that  $\langle -, - \rangle, \pi_1, \pi_2 \notin \Sigma$ .

**Reductions.** Let  $\rightarrow_R$  be a binary relation on  $\Lambda(\Sigma)$ .

(i) We denote by  $\rightarrow_R^+$  the transitive closure of  $\rightarrow_R$ , by  $\rightarrow_R^*$  its reflexive and transitive closure, and by  $\leftarrow_R$  its inverse. Moreover, we write  $t \rightarrow_R^k u$  if  $t \rightarrow_R^* u$  in less than  $k \in \mathbb{N}$  steps.

(ii) Given  $A \subseteq \Lambda(\Sigma)$ , we let

$$\begin{aligned} (A)_R &=_{\text{def}} \{t \mid \exists u. u \in A \wedge t \rightarrow_R u\} \\ \text{and} \quad (A)_R^* &=_{\text{def}} \{t \mid \exists u. u \in A \wedge t \rightarrow_R^* u\}. \end{aligned}$$

Given  $t \in \Lambda(\Sigma)$ , we write  $(t)_R$  for  $(\{t\})_R$  and  $(t)_R^*$  for  $(\{t\})_R^*$ . We say that  $t$  is *R-reducible* (or *reducible*) if  $(t)_R \neq \emptyset$  and that it is *R-normal* (or *normal*) otherwise.

(iii) We denote by  $\mathcal{SN}_R$  the set of *strongly R-normalizing* (or *strongly-normalizing*) terms for  $\rightarrow_R$ , which is the smallest set of terms such that

$$\forall t. (\forall u. t \rightarrow_R u \implies u \in \mathcal{SN}_R) \implies t \in \mathcal{SN}_R.$$

(iv) We define the *product extension* of  $\rightarrow_R$  as  $(t_1, \dots, t_n) \rightarrow_R (u_1, \dots, u_n)$  when there is  $k \in \{1, \dots, n\}$  such that  $t_k \rightarrow_R u_k$  and  $t_i = u_i$  for all  $i \neq k$ .

**Definition 2.1.1 (Rewrite Relation)** Let  $\rightarrow_R$  be a relation on  $(\Lambda(\Sigma) \setminus \mathcal{X}) \times \Lambda(\Sigma)$ .

We say that  $\rightarrow_R$  is *closed by substitutions* if  $t \rightarrow_R u$  implies  $t\sigma \rightarrow_R u\sigma$ , and that it is *closed by contexts* if it is closed by the following rules:

$$\frac{t \rightarrow_R u}{\lambda x. t \rightarrow_R \lambda x. u} \quad \frac{(t_1, t_2) \rightarrow_R (u_1, u_2)}{t_1 t_2 \rightarrow_R u_1 u_2} \quad \frac{(t_1, \dots, t_n) \rightarrow_R (u_1, \dots, u_n)}{f(t_1, \dots, t_n) \rightarrow_R f(u_1, \dots, u_n)}$$

where  $f \in \Sigma \uplus \{\langle -, - \rangle, \pi_1, \pi_2\}$ . We say that  $\rightarrow_R$  is a *rewrite relation* on  $\Lambda(\Sigma)$  if it is closed by contexts and substitutions.

In the following,  $\rightarrow_R$  denotes a rewrite relation on  $\Lambda(\Sigma)$ . Note that variables  $x \in \mathcal{X}$  are always in *R-normal* form. Given two relations  $\rightarrow_A$  and  $\rightarrow_B$ , we write  $\rightarrow_{AB}$  for  $\rightarrow_A \cup \rightarrow_B$ .

The reduction relations  $\rightarrow_\beta$  of the  $\lambda$ -calculus and  $\rightarrow_\pi$  of products are the least rewrite relations on  $\Lambda(\Sigma)$  containing respectively  $\mapsto_\beta$  and  $\mapsto_\pi$ , where

$$(\lambda x. t)u \mapsto_\beta t[u/x] \quad \text{and} \quad \pi_1 \langle t_1, t_2 \rangle \mapsto_\pi t_1 \quad \pi_2 \langle t_1, t_2 \rangle \mapsto_\pi t_2.$$

**Rewriting.** A rewrite system  $\mathcal{R}$  on  $\Lambda(\Sigma)$  is a set of pairs written

$$f(l_1, \dots, l_n) \mapsto_{\mathcal{R}} r ,$$

where  $r \in \Lambda(\Sigma)$  and for all  $i \in \{1, \dots, n\}$ ,  $l_i$  is an *algebraic term*, that is, a term on the grammar

$$p ::= x \mid g(p_1, \dots, p_m) \quad \text{where} \quad g \in \Sigma_m .$$

The reduction relation  $\rightarrow_{\mathcal{R}}$  issued from a rewrite system  $\mathcal{R}$  is the least rewrite relation on  $\Lambda(\Sigma)$  containing  $\mapsto_{\mathcal{R}}$ . In this paper, we also consider some particular cases of rewriting, namely *constructor* rewriting and *orthogonal* rewriting.

**Definition 2.1.2 (Constructor Rewriting)** Assume given a set  $\mathcal{C} \subseteq \Sigma$  of constructor symbols  $c$ . A constructor rewrite system with constructors in  $\mathcal{C}$  is a rewrite system  $\mathcal{R}$  such that for each rule  $f(l_1, \dots, l_n) \mapsto_{\mathcal{R}} r$  and all  $i \in \{1, \dots, n\}$ , the term  $l_i$  is a pattern, that is, a term on the grammar

$$p ::= x \mid c(p_1, \dots, p_n) \quad \text{where} \quad c \in \mathcal{C} .$$

Orthogonal rewriting is a widespread, well-understood form of rewriting [Ter03].

**Definition 2.1.3 (Orthogonal Rewriting)** A rewrite system  $\mathcal{R}$  is orthogonal if there is no superpositions of rewrite rules and if all rules are left-linear.

A logically powerful orthogonal constructor rewrite system, realizing Spector's double negation shift, is studied in [CS06]. It is originally presented in curried form, but can also be written in our framework with algebraic symbols of fixed arity. For the purpose of this paper, it is convenient to have in mind a very simple system.

**Example 2.1.4** Peano's numbers are build using the constructors 0 (of arity 0) and S (of arity 1). The following system, defining addition, is an orthogonal constructor rewrite system:

$$\text{plus}(x, 0) \mapsto_{\text{plus}} x \quad \text{plus}(x, S(y)) \mapsto_{\text{plus}} \text{plus}(S(x), y) .$$

**Typing.** Given base types  $B \in \mathcal{B}$ , simple types with products are defined as usual:

$$T, U \in \mathcal{T}_{\Rightarrow \times}(\mathcal{B}) ::= B \mid U \Rightarrow T \mid T \times U .$$

We denote by  $\mathcal{T}_{\Rightarrow}(\mathcal{B})$  (resp.  $\mathcal{T}_{\times}(\mathcal{B})$ ) the set of pure simple types (resp. pure products types), which are types  $T \in \mathcal{T}_{\Rightarrow \times}(\mathcal{B})$  with no occurrence of  $\times$  (resp.  $\Rightarrow$ ).

*Typing contexts* are functions  $\Gamma$  of finite domain from  $\mathcal{X}$  to  $\mathcal{T}_{\Rightarrow \times}(\mathcal{B})$  written  $x_1 : T_1, \dots, x_n : T_n$ . The typing relation  $\Gamma \vdash_{\Rightarrow \times} t : T$  of the  $\lambda$ -calculus with products is the least relation closed under the following rules:

$$\begin{aligned} & (\text{AX}) \frac{}{\Gamma, x : T \vdash x : T} \\ & (\Rightarrow \text{I}) \frac{\Gamma, x : U \vdash t : T}{\Gamma \vdash \lambda x. t : U \Rightarrow T} \quad (\Rightarrow \text{E}) \frac{\Gamma \vdash t : U \Rightarrow T \quad \Gamma \vdash u : U}{\Gamma \vdash t u : T} \\ & (\times \text{E}) \frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash \pi_i t : T_i} \quad (i \in \{1, 2\}) \quad (\times \text{I}) \frac{\Gamma \vdash t_1 : T_1 \quad t_2 : T_2}{\Gamma \vdash \langle t_1, t_2 \rangle : T_1 \times T_2} \end{aligned}$$

The typing relation  $\Gamma \vdash_{\Rightarrow} t : T$  of the pure  $\lambda$ -calculus (resp.  $\Gamma \vdash_{\times} t : T$  of the pure product calculus) is the restriction of  $\vdash_{\Rightarrow \times}$  to contexts  $\Gamma : \mathcal{X} \mapsto \mathcal{T}_{\Rightarrow}(\mathcal{B})$  and to types  $T \in \mathcal{T}_{\Rightarrow}(\mathcal{B})$  (resp.  $\Gamma : \mathcal{X} \mapsto \mathcal{T}_{\times}(\mathcal{B})$  and  $T \in \mathcal{T}_{\times}(\mathcal{B})$ ).

Given a *type assignment*  $\tau : \prod_{n \in \mathbb{N}} \Sigma_n \mapsto \mathcal{P}(\mathcal{T}_{\Rightarrow \times}(\mathcal{B})^{n+1})$ , the typing relation  $\Gamma \vdash_{\Rightarrow \times \tau}$  of the  $\lambda$ -calculus with products and algebraic symbols typed in  $(\Sigma, \tau)$  is the least relation closed under the rules of  $\vdash_{\Rightarrow \times}$  and the rule

$$(\text{SYMB}) \frac{\Gamma \vdash t_1 : T_1 \quad \dots \quad \Gamma \vdash t_n : T_n}{\Gamma \vdash f(t_1, \dots, t_n) : T} \quad (T_1, \dots, T_n, T) \in \tau(f)$$

The  $\lambda$ -calculus with typed algebraic symbols is obtained from  $\tau : \prod_{n \in \mathbb{N}} \Sigma_n \mapsto \mathcal{P}(\mathcal{T}_{\Rightarrow}(\mathcal{B})^{n+1})$ . Its typing relation  $\vdash_{\Rightarrow \tau}$  is the least relation closed under the rules of  $\vdash_{\Rightarrow}$  and the rule (SYMB).

In the following, we denote by  $\mathcal{T}_{\text{ty}}(\mathcal{B})$  an arbitrary set of types build from base types  $B \in \mathcal{B}$ , arrow types  $\Rightarrow$  and product types  $\times$ , and by  $\vdash_{\text{ty}}$  an arbitrary type system obtained by any combination of the pure  $\lambda$ -calculus, products and typed algebraic symbols.

**Typed rewriting.** Given  $\tau : \prod_{n \in \mathbb{N}} \Sigma_n \mapsto \mathcal{P}(\mathcal{T}_{\text{ty}}(\mathcal{B})^{n+1})$ , a rewrite system  $\mathcal{R}$  is *typed in*  $\vdash_{\text{ty}} \tau$  if for all  $f \in \Sigma$ , all  $(T_1, \dots, T_n, T) \in \tau(f)$  and for each rewrite rule  $f(\vec{l}) \mapsto_{\mathcal{R}} r$  there exists a (necessarily unique) context  $\Gamma$  with  $\text{Dom}(\Gamma) = \text{FV}(f(\vec{l}))$  such that

$$\Gamma \vdash_{\text{ty}} f(\vec{l}) : T \quad \text{and} \quad \Gamma \vdash_{\text{ty}} r : T.$$

A *constructor* rewrite system is typed if it is typed as a rewrite system and if moreover, for all  $c \in \mathcal{C}$ ,  $\tau(c)$  is of the form  $\{(\vec{T}, B)\}$  with  $B \in \mathcal{B}$ .

**Example 2.1.5 (Ex. 2.1.4 continued)** *The structure of Peano's numbers can be typed using the base type  $\text{Nat} \in \mathcal{B}$ , by putting  $\tau(0) = \{\text{Nat}\}$  and  $\tau(S) = \{(\text{Nat}, \text{Nat})\}$ . If moreover  $\tau(\text{plus}) = \{(\text{Nat}, \text{Nat}, \text{Nat})\}$ , then the rewrite system  $\mapsto_{\text{plus}}$  is a typed constructor rewrite system.*

## 2.2 Reducibility

Given a type system  $\vdash_{\text{ty}}$  and a rewrite relation  $\rightarrow_{\mathcal{R}}$ , we are interested in properties of some methods to prove that typable terms are strongly  $\mathcal{R}$ -normalizing.

**Type interpretations.** The methods we are interested in are based on *type interpretations* which map types  $T$  to sets of strongly normalizing terms  $\llbracket T \rrbracket$ . A type interpretation can be used to prove strong normalization when it is *adequate*, that is when

$$\vdash_{\text{ty}} t : T \quad \text{implies} \quad t \in \llbracket T \rrbracket.$$

**Definition 2.2.1 (Type Interpretation)** *Let  $\rightarrow_{\mathcal{R}}$  be a rewrite relation on  $\Lambda(\Sigma)$ .*

(i) *A type interpretation is a map  $\llbracket \_ \rrbracket : \mathcal{T}_{\text{ty}}(\mathcal{B}) \mapsto \mathcal{P}(\Lambda(\Sigma))$  such that  $\mathcal{X} \subseteq \llbracket T \rrbracket \subseteq \mathcal{SN}_{\mathcal{R}}$  for all  $T \in \mathcal{T}_{\text{ty}}(\mathcal{B})$ .*

(ii) *A type interpretation  $\llbracket \_ \rrbracket$  is adequate for a type system  $\vdash_{\text{ty}}$  if*

$$(\Gamma \vdash_{\text{ty}} t : T \quad \wedge \quad \sigma \models_{\llbracket \_ \rrbracket} \Gamma) \implies t\sigma \in \llbracket T \rrbracket,$$

*where  $\sigma \models_{\llbracket \_ \rrbracket} \Gamma$  iff  $\sigma(x) \in \llbracket \Gamma(x) \rrbracket$  for all  $x \in \text{Dom}(\Gamma)$ .*



If we have a type interpretation  $\llbracket \_ \rrbracket$  which is adequate for a type system  $\vdash_{\text{ty}}$ , then every term typable in  $\vdash_{\text{ty}}$  is strongly normalizing.

**Proposition 2.2.2 (Strong Normalization)** *Let  $\rightarrow_{\mathcal{R}}$  be a rewrite relation on  $\Lambda(\Sigma)$  and  $\vdash_{\text{ty}}$  be a type system. If  $\llbracket \_ \rrbracket : \mathcal{T}_{\text{ty}}(\mathcal{B}) \mapsto \mathcal{P}(\mathcal{SN}_{\mathcal{R}})$  is a type interpretation which is adequate for  $\vdash_{\text{ty}}$ , then*

$$\Gamma \vdash_{\text{ty}} t : T \implies t \in \mathcal{SN}_{\mathcal{R}}.$$

PROOF. Define the substitution  $\sigma =_{\text{def}} [x/x \mid x \in \text{Dom}(\Gamma)]$ . Since  $\llbracket \_ \rrbracket$  is a type interpretation, we have  $\mathcal{X} \subseteq \llbracket T \rrbracket$  for all  $T \in \mathcal{T}_{\text{ty}}(\mathcal{B})$ , hence  $\sigma \models_{\llbracket \_ \rrbracket} \Gamma$ . Since  $\llbracket \_ \rrbracket$  is adequate for  $\vdash_{\text{ty}}$ , we have  $t\sigma = t \in \llbracket T \rrbracket$ , hence  $t \in \mathcal{SN}_{\mathcal{R}}$  since  $\llbracket \_ \rrbracket$  is a type interpretation.  $\square$

**Closure operators.** Adequacy requires interpretations  $\llbracket \_ \rrbracket$  to satisfy some closure properties. It turns out that *closure operators* provide a convenient level of abstraction to formulate and reason on these properties. Our use of closure operators is inspired from [Vou04, VM04]. We recall here some facts on this notion.

Let  $\mathcal{D} = (\mathcal{D}, \leq, \bigvee, \bigwedge, \perp, \top)$  be a complete lattice.

**Definition 2.2.3 (Closure Operator)** *A closure operator on  $\mathcal{D}$  is a function  $\overline{\_} : \mathcal{D} \mapsto \mathcal{D}$  which is idempotent:  $\overline{\overline{d}} = \overline{d}$  for all  $d \in \mathcal{D}$ ; extensive:  $d \leq \overline{d}$  for all  $d \in \mathcal{D}$ ; and monotone:  $d \leq e$  implies  $\overline{d} \leq \overline{e}$  for all  $d, e \in \mathcal{D}$ .*

*An element  $d \in \mathcal{D}$  is closed for  $\overline{\_}$  if  $\overline{d} = d$ . We denote by  $\overline{\mathcal{D}}$  the set of closed elements of  $\mathcal{D}$ .*

**Proposition 2.2.4** *An element  $d \in \mathcal{D}$  is closed for  $\overline{\_}$  if and only if  $d = \overline{e}$  for some  $e \in \mathcal{D}$ .*

PROOF. The "only if" direction is trivial. If  $d = \overline{e}$ , then  $\overline{d} = \overline{\overline{e}} = \overline{e} = d$  by idempotence.  $\square$

Closure operators preserve greatest lower bounds.

**Proposition 2.2.5**  $X \subseteq \overline{\mathcal{D}} \implies \bigwedge X \in \overline{\mathcal{D}}$ .

PROOF. Given  $X \subseteq \overline{\mathcal{D}}$ , we show that  $\overline{\bigwedge X} = \bigwedge X$ . Since  $\overline{\_}$  is extensive, it suffices to show that  $\overline{\bigwedge X} \leq \bigwedge X$ . By definition of  $\bigwedge$ , this holds if  $\overline{\bigwedge X} \leq d$  for all  $d \in X$ . But if  $d \in X$ , since  $\bigwedge X \leq d$ , by monotony of  $\overline{\_}$  we deduce  $\overline{\bigwedge X} \leq \overline{d}$ , hence  $\overline{\bigwedge X} \leq d$  because  $d$  is closed.  $\square$

It follows that the set of closed element of  $\mathcal{D}$  is also a complete lattice. But note that its least upper bounds may not be the least upper bounds of  $\mathcal{D}$ . Let  $\overline{\bigvee X} =_{\text{def}} \overline{\bigvee X}$  for all  $X \subseteq \overline{\mathcal{D}}$ .

**Lemma 2.2.6**  $\overline{\mathcal{D}} =_{\text{def}} (\overline{\mathcal{D}}, \leq, \overline{\bigvee}, \bigwedge, \perp, \top)$  is a complete lattice.

PROOF. First, since  $\overline{\_}$  is extensive, we have  $\top \leq \overline{\top}$ , hence  $\overline{\top} = \top$ . By monotony of  $\overline{\_}$ , for all  $d \in \mathcal{D}$  we have  $\perp \leq \overline{d} \leq \top$ . Moreover, by Prop. 2.2.5, the g.l.b.'s of  $\overline{\mathcal{D}}$  are given by  $\bigwedge$ .

Let us now show that  $\overline{\bigvee}$  gives the l.u.b.'s of  $\overline{\mathcal{D}}$ . Let  $X \subseteq \overline{\mathcal{D}}$ . We have  $\bigvee X \leq \overline{\bigvee X}$  by extensivity, hence  $\overline{\bigvee X}$  is an upper bound of  $X$ . We now show that it is the least upper bound of  $X$ : if  $e \in \overline{\mathcal{D}}$  is such that  $d \leq e$  for all  $d \in X$ , then  $\overline{\bigvee X} \leq e$ . But if  $d \leq e$  for all  $d \in X$ , then  $\bigvee X \leq e$  by definition of  $\bigvee$ , hence  $\overline{\bigvee X} \leq \overline{e}$  by monotony. It follows that  $\overline{\bigvee X} \leq e$  because  $e$  is closed.  $\square$

In particular, given a closure operator  $\overline{\_} : \mathcal{P}(\mathcal{D}) \mapsto \mathcal{P}(\mathcal{D})$ , the greatest element of the complete lattice  $\mathcal{P}(\overline{\mathcal{D}})$  is  $\mathcal{D}$  and its g.l.b.'s are given by intersections.

**Reducibility families.** Using closure operators, we can study and compare general properties of interpretations  $\llbracket \_ \rrbracket$ , by opposition to properties of particular instances  $\llbracket T \rrbracket$ . Therefore, we assume that a type interpretation  $\llbracket \_ \rrbracket$  is a map from types to a set of subsets of  $\mathcal{SN}_R$  called a *reducibility family*. Using the notion of reducibility family is even mandatory when dealing with impredicative type systems such as system F [Gir72, GLT89, Gal89].

Recall that by definition of a rewrite relation, variables  $x \in \mathcal{X}$  are always in normal form.

**Definition 2.2.7 (Reducibility Family)** *Let  $\rightarrow_R$  be a rewrite relation on  $\Lambda(\Sigma)$ .*

- (i) *A reducibility family for  $\rightarrow_R$  is a collection of sets  $\mathcal{Red}$  issued from a closure operator  $\mathcal{Red} : \mathcal{P}(\mathcal{SN}_R) \mapsto \mathcal{SN}(\mathcal{SN}_R)$  such that  $\mathcal{X} \subseteq \mathcal{Red}(X)$  for all  $X \subseteq \mathcal{SN}_R$ .*
- (ii) *A type interpretation in  $\mathcal{Red}$  is a type interpretation which is a map  $\llbracket \_ \rrbracket : \mathcal{T}_{ty}(\mathcal{B}) \mapsto \mathcal{Red}$ .*

By assuming that  $\mathcal{Red}$  is given by a closure operator, we know that it is a complete lattice whose g.l.b. are given by intersections. According to Prop. 2.2.2, if we have a type interpretation  $\llbracket \_ \rrbracket$  which is adequate for a type system  $\vdash_{ty}$ , then every term typable in  $\vdash_{ty}$  is strongly normalizing.

**Proposition 2.2.8 (Strong Normalization)** *Let  $\rightarrow_R$  be a rewrite relation on  $\Lambda(\Sigma)$ ,  $\vdash_{ty}$  be a type system and  $\mathcal{Red}$  be a reducibility family for  $\rightarrow_R$ . If  $\llbracket \_ \rrbracket : \mathcal{T}_{ty}(\mathcal{B}) \mapsto \mathcal{Red}$  is a type interpretation which is adequate for  $\vdash_{ty}$ , then*

$$\Gamma \vdash_{ty} t : T \implies t \in \mathcal{SN}_R.$$

### 3 Toward Saturated Sets

In this section, we analyze some requirements on a type interpretation  $\llbracket \_ \rrbracket$  in order to satisfy the conditions of Def. 2.2.1. Namely

- $\mathcal{X} \subseteq \llbracket T \rrbracket \subseteq \mathcal{SN}_R$  for all  $T \in \mathcal{T}_{ty}(\mathcal{B})$ ,
- if  $\Gamma \vdash_{ty} t : T$  and  $\sigma \models_{\llbracket \_ \rrbracket} \Gamma$  then  $t\sigma \in \llbracket T \rrbracket$ .

We consider the pure  $\lambda$ -calculus, the  $\lambda$ -calculus with binary products, and the combination of  $\lambda$ -calculus with rewriting. As we will see, in first two cases we are naturally lead to Tait's *saturated sets* [Tai75]. The main characteristic of these sets is that types interpretations  $\llbracket T \rrbracket$  are closed by strong-normalization-preserving *weak-head expansions*. However, the situation is more complex with rewriting. Indeed, we will see in Ex. 3.5.1 that some rewrite system do not admit any notion of weak standardization preserving strong normalization. This prevent us from defining a general natural notion of Tait's saturated sets for rewriting.

Our aim is to isolate and analyze some elementary computational properties on which saturated sets rely, and to precisely see when and how they are used in the proofs of basic and well-known results on saturated sets. This leads to the notion of non-interaction, on which our reducibility candidates of Sect. 4 are based. The content of this section was briefly sketched in [Rib08].

#### 3.1 Basic Mechanisms with Products

We begin by recalling some basic mechanisms of reducibility. We concentrate on a very simple system which only features product types. Our plan is to see some sufficient conditions to

ensure that a type interpretation  $\llbracket \_ \rrbracket$  is adequate. For the sake of simplicity, typing contexts are leaved implicit.

The rules

$$(\times E_0) \frac{t : T_1 \times T_2}{\pi_i t : T_i} \ (i \in \{1, 2\}) \quad (\times I_0) \frac{t_1 : T_1 \quad t_2 : T_2}{\langle t_1, t_2 \rangle : T_1 \times T_2}$$

impose that

$$\begin{aligned} & \text{if } t \in \llbracket T_1 \rrbracket \times \llbracket T_2 \rrbracket \text{ then } \pi_1 t \in \llbracket T_1 \rrbracket \text{ and } \pi_2 t \in \llbracket T_2 \rrbracket, & (\times E_0) \\ & \text{if } t_1 \in \llbracket T_1 \rrbracket \text{ and } t_2 \in \llbracket T_2 \rrbracket \text{ then } \langle t_1, t_2 \rangle \in \llbracket T_1 \rrbracket \times \llbracket T_2 \rrbracket; & (\times I_0) \end{aligned}$$

that is

$$\{\langle t, u \rangle \mid t \in \llbracket T_1 \rrbracket \wedge u \in \llbracket T_2 \rrbracket\} \subseteq \llbracket T_1 \rrbracket \times \llbracket T_2 \rrbracket \subseteq \{t \mid \pi_1 t \in \llbracket T_1 \rrbracket \wedge \pi_2 t \in \llbracket T_2 \rrbracket\}. \quad (1)$$

Property (1) leads to

$$\forall t, u. \ (t \in \llbracket T_1 \rrbracket \wedge u \in \llbracket T_2 \rrbracket) \implies (\pi_1 \langle t, u \rangle \in \llbracket T_1 \rrbracket \wedge \pi_2 \langle t, u \rangle \in \llbracket T_2 \rrbracket).$$

In particular, if we interpret base types  $B \in \mathcal{B}$  by  $\mathcal{SN}_\pi$ , we need that for all  $T \in \mathcal{T}_\times(\mathcal{B})$ ,

$$\forall t, u. \ (t \in \llbracket T \rrbracket \wedge u \in \mathcal{SN}_\pi) \implies (\pi_1 \langle t, u \rangle \in \llbracket T \rrbracket \wedge \pi_2 \langle u, t \rangle \in \llbracket T \rrbracket). \quad (2)$$

In words,  $\llbracket T \rrbracket$  has to be closed by strong-normalization preserving weak head expansion. Let us now see how we can ensure this property for all  $T \in \mathcal{T}_\times(\mathcal{B})$ . We reason by induction on  $T \in \mathcal{T}_\times(\mathcal{B})$ , assuming that  $\_ \times \_$  is a function from  $\mathcal{P}(\Lambda(\Sigma_\pi))^2$  to  $\mathcal{P}(\Lambda(\Sigma_\pi))$  satisfying (1).

$T = B \in \mathcal{B}$ . We must show that

$$\forall t, u \in \mathcal{SN}_\pi. \ \pi_1 \langle t, u \rangle \in \mathcal{SN}_\pi \wedge \pi_2 \langle t, u \rangle \in \mathcal{SN}_\pi. \quad (3)$$

$T = T_1 \times T_2$ . We must show that for all  $t \in \llbracket T_1 \rrbracket \times \llbracket T_2 \rrbracket$  and all  $u \in \mathcal{SN}_\pi$ ,

$$\pi_1 \pi_1 \langle t, u \rangle, \pi_1 \pi_2 \langle u, t \rangle \in \llbracket T_1 \rrbracket \wedge \pi_2 \pi_1 \langle t, u \rangle, \pi_2 \pi_2 \langle u, t \rangle \in \llbracket T_2 \rrbracket. \quad (4)$$

The induction on types imposes, *via* (4), that (2) is satisfied for terms placed inside contexts of the form  $\pi_{i_1} \cdots \pi_{i_n} \_$ . These contexts correspond, at the term level, to *elimination* rules of product types. We call them *elimination contexts*. Our use of elimination contexts is inspired from [Abe04, Mat05]. In the case of products, they are given by the following abstract syntax:

$$E[\_] \in \mathcal{E}_\times \quad ::= \quad [\_] \mid \pi_1 E[\_] \mid \pi_2 E[\_].$$

If we write (2) using elimination contexts as in (4), then we get the following property: for all  $T \in \mathcal{T}_\times(\mathcal{B})$ , all  $t_1, t_2$ , and all  $E[\_] \in \mathcal{E}_\times$ ,

$$\begin{aligned} (t_2 \in \mathcal{SN}_\pi \wedge E[t_1] \in \llbracket T \rrbracket) & \implies E[\pi_1 \langle t_1, t_2 \rangle] \in \llbracket T \rrbracket, \\ (t_1 \in \mathcal{SN}_\pi \wedge E[t_2] \in \llbracket T \rrbracket) & \implies E[\pi_2 \langle t_1, t_2 \rangle] \in \llbracket T \rrbracket. \end{aligned} \quad (5)$$

In other words, the interpretation of types must be stable by elimination contexts.

**Remark 3.1.1** *The relation  $\{(E[t], E[u]) \mid E[\_] \in \mathcal{E}_\times \wedge t \mapsto_\pi u\}$  corresponds for products to the weak-head  $\beta$ -reduction of the  $\lambda$ -calculus (see also Rem. 3.2.3).*

### 3.2 Pure Lambda-Calculus

We are now going to use the intuitions for the products presented in the preceding section to define saturated sets for the simply typed  $\lambda$ -calculus. The material presented here can be found in a slightly different form in [Kri90, Bar92, GLT89]. We focus on the rules  $(\Rightarrow E)$  and  $(\Rightarrow I)$ :

$$(\Rightarrow E) \frac{\Gamma \vdash t : U \Rightarrow T \quad \Gamma \vdash u : U}{\Gamma \vdash tu : T} \quad (\Rightarrow I) \frac{\Gamma, x : U \vdash t : T}{\Gamma \vdash \lambda x. t : U \Rightarrow T}$$

Reasoning as in (1), we obtain that  $\_ \Rightarrow \_$  must satisfy, for all substitution  $\sigma$  such that  $\sigma \models \Gamma$ ,

$$\begin{aligned} \{(\lambda x. t)\sigma \mid \forall u. u \in \llbracket U \rrbracket \implies t\sigma[u/x] \in \llbracket T \rrbracket\} \\ \subseteq \llbracket U \rrbracket \Rightarrow \llbracket T \rrbracket \subseteq \\ \{t\sigma \mid \forall u. u \in \llbracket U \rrbracket \implies t\sigma u \in \llbracket T \rrbracket\} . \end{aligned} \quad (6)$$

There are different possible definitions of  $\_ \Rightarrow \_$  satisfying (6). We use an interpretation based on eliminations, in the sense of [Mat98].

**Definition 3.2.1 (Function Space)** Define  $\_ \Rightarrow \_ : \mathcal{P}(\Lambda(\Sigma)) \times \mathcal{P}(\Lambda(\Sigma)) \mapsto \mathcal{P}(\Lambda(\Sigma))$  as

$$A \Rightarrow B \quad =_{\text{def}} \quad \{t \mid \forall u. u \in A \implies tu \in B\} .$$

We now define the interpretation of simples types  $T \in \mathcal{T}_{\Rightarrow}(\mathcal{B})$ . As in Sect. 3.1, we interpret base types by  $\mathcal{SN}_{\beta}$ .

**Definition 3.2.2 (Type Interpretation)** The interpretation of a type  $T \in \mathcal{T}_{\Rightarrow}(\mathcal{B})$  is the set  $\llbracket T \rrbracket$  defined by induction on  $T$  as follows:

$$\begin{aligned} \llbracket B \rrbracket &=_{\text{def}} \mathcal{SN}_{\beta} & \text{if } B \in \mathcal{B} , \\ \llbracket U \Rightarrow T \rrbracket &=_{\text{def}} \llbracket U \rrbracket \Rightarrow \llbracket T \rrbracket . \end{aligned}$$

We are now going to show that the interpretation  $\llbracket \_ \rrbracket$  is adequate. With the same reasoning as in (2), it follows from (6) that for all  $T \in \mathcal{T}_{\Rightarrow}(\mathcal{B})$ ,

$$\forall t, u. u \in \mathcal{SN}_{\beta} \implies (t[u/x] \in \llbracket T \rrbracket \implies (\lambda x. t)u \in \llbracket T \rrbracket) . \quad (7)$$

**Remark 3.2.3** The relation  $\{(E[t], E[u]) \mid E[\_] \in \mathcal{E}_{\Rightarrow} \wedge t \mapsto_{\beta} u\}$  is the weak-head  $\beta$ -reduction. Hence, type interpretations are closed by weak-head  $\beta$ -expansion. Weak head reduction is the main notion of the Krivine's Abstract Machine, and stability by weak-head expansion is the main property required by truth values of [Kri04].

In order to show that  $\llbracket \_ \rrbracket$  is a type interpretation, we must show that  $\mathcal{X} \subseteq \llbracket T \rrbracket$  and  $\llbracket T \rrbracket \subseteq \mathcal{SN}_{\beta}$  for all  $T \in \mathcal{T}_{\Rightarrow}(\mathcal{B})$ . These two properties are not independent from each other. Indeed, knowing that  $\llbracket U \rrbracket, \llbracket T \rrbracket \subseteq \mathcal{SN}_{\beta}$  is not enough to ensure that  $\llbracket U \rrbracket \Rightarrow \llbracket T \rrbracket \subseteq \mathcal{SN}_{\beta}$  because

$$\emptyset \Rightarrow \llbracket T \rrbracket = \{t \mid \forall u. u \in \emptyset \implies tu \in \llbracket T \rrbracket\} = \Lambda(\Sigma) . \quad (8)$$

Thus we can have  $\llbracket U \rrbracket \Rightarrow \llbracket T \rrbracket \subseteq \mathcal{SN}_{\beta}$  only if  $\llbracket U \rrbracket \neq \emptyset$ . This is ensured by  $\mathcal{X} \subseteq \llbracket U \rrbracket$ . On the other hand, in order to show that  $\mathcal{X} \subseteq \llbracket U \rrbracket \Rightarrow \llbracket T \rrbracket$ , we have to show that  $xu \in \llbracket T \rrbracket$  for all  $u \in \llbracket U \rrbracket$  and all  $x \in \mathcal{X}$ . Hence, assuming that  $\llbracket T \rrbracket \subseteq \mathcal{SN}_{\beta}$ , we must have  $\llbracket U \rrbracket \subseteq \mathcal{SN}_{\beta}$  in order to have  $\mathcal{X} \subseteq \llbracket U \rrbracket \Rightarrow \llbracket T \rrbracket$ . To summarize, we need  $\mathcal{X} \subseteq \llbracket U \rrbracket \subseteq \mathcal{SN}_{\beta}$  for all type  $U$ .

**Remark 3.2.4** Note that when  $\mathbb{T}$  occurs on the left of an arrow, the property  $\mathcal{X} \subseteq \llbracket \mathbb{T} \rrbracket \subseteq \mathcal{SN}_\beta$  is needed because we use reducibility to prove strong normalization. In other frameworks, and if we are interested in other properties than strong normalization, for instance as in [Kri04], it may happen that no condition is imposed on the interpretation of the left argument of an arrow.

The property  $\mathcal{X} \subseteq \llbracket \mathbb{T} \rrbracket \subseteq \mathcal{SN}_\beta$  must be satisfied for bases types, but also for the function space. This leads us to formulate them using elimination contexts, as in (5).

**Definition 3.2.5 (Elimination Contexts)** Let  $\Sigma$  be a signature. The set  $\mathcal{E}_\Rightarrow$  is generated by the following grammar:

$$E[\ ] \in \mathcal{E}_\Rightarrow \quad ::= \quad [\ ] \quad | \quad E[\ ] \, t \quad \text{where} \quad t \in \Lambda(\Sigma) .$$

Note that  $E[t] \in \mathcal{SN}_\beta$  implies  $t \in \mathcal{SN}_\beta$ . If we formulate property (7) and  $\mathcal{X} \subseteq \llbracket \mathbb{T} \rrbracket \subseteq \mathcal{SN}_\beta$  using elimination contexts, then we get that types must be interpreted by sets  $A \subseteq \mathcal{SN}_\beta$  such that for all  $E[\ ] \in \mathcal{E}$ , all  $x \in \mathcal{X}$  and all  $t, u \in \Lambda$ ,

$$E[\ ] \in \mathcal{SN}_\beta \quad \implies \quad E[x] \in A , \quad (9)$$

$$(E[t[u/x]] \in A \quad \wedge \quad u \in \mathcal{SN}_\beta) \quad \implies \quad E[(\lambda x. t)u] \in A . \quad (10)$$

The sets  $A \subseteq \mathcal{SN}_\beta$  satisfying (9) and (10) are *Tait's saturated sets* [Tai75] (see also [Kri90, Bar92, Gal89]).

**Definition 3.2.6 (Saturated Sets)** The set  $\mathcal{SAT}_\beta$  of  $\beta$ -saturated sets is the set of all  $S \subseteq \mathcal{SN}_\beta$  such that

(SAT1) if  $E[\ ] \in \mathcal{SN}_\beta$  and  $x \in \mathcal{X}$  then  $E[x] \in S$ ,

(SAT2 $_\beta$ ) if  $E[t[u/x]] \in S$  and  $u \in \mathcal{SN}_\beta$  then  $E[(\lambda x. t)u] \in S$ .

In order to ensure that Def. 3.2.6 makes sense, we have to show that  $\mathcal{SAT}_\beta$  is not empty. This amounts to showing that  $\mathcal{SN}_\beta \in \mathcal{SAT}_\beta$ . We must check properties (9) and (10) with  $A = \mathcal{SN}_\beta$ , which in this case are consequences of two important facts.

First, a reduction step from a term of the form  $E[(\lambda x. t)u]$  (resp.  $E[x]$ ) occurs either in the elimination context  $E[\ ]$  or in the term  $(\lambda x. t)u$ , but involves no interaction between them. This is expressed by the following obvious lemma.

**Lemma 3.2.7 (Non-Interaction)**

$$E[x] \rightarrow_\beta v \quad \implies \quad (v = E'[x] \quad \text{with} \quad E[\ ] \rightarrow_\beta E'[\ ]) \quad (11)$$

$$E[(\lambda x. t)u] \rightarrow_\beta v \quad \implies \quad (v = E'[s] \quad \text{with} \quad (E[\ ], (\lambda x. t)u) \rightarrow_\beta (E'[\ ], s)) \quad (12)$$

Second, property (10) follows from property (12) and the fact that  $(\lambda x. t)u \in \mathcal{SN}_\beta$  as soon as  $t[u/x] \in \mathcal{SN}_\beta$  and  $u \in \mathcal{SN}_\beta$ . This property holds in turn thanks to the *Weak Standardization Lemma*, which was used in [Alt93] for extensions of the Calculus of Constructions. It is obvious for the pure  $\lambda$ -calculus.

**Lemma 3.2.8 (Weak Standardization)**

$$(\lambda x. t)u \rightarrow_\beta v \quad \implies \quad (v = t[u/x] \quad \text{or} \quad v = (\lambda x. t')u' \quad \text{with} \quad (t, u) \rightarrow_\beta (t', u'))$$

PROOF. Since  $E[\ ]$  is of the form  $[\ ]t_1 \dots t_n$ , the property is

$$\begin{array}{ccc} (\lambda x.t)u t_1 \dots t_n & \xrightarrow{\beta} & t[u/x]t_1 \dots t_n \\ \beta \downarrow & & \downarrow * \beta \\ (\lambda x.t')u' t'_1 \dots t'_n & \xrightarrow{\beta} & t'[u'/x]t'_1 \dots t'_n \end{array}$$

□

We can now show that  $\mathcal{SN}_\beta \in \mathcal{SAT}_\beta$ . The proof scheme used for the clause  $(\mathcal{SAT2}_\beta)$ , which relies on non-interaction and on weak standardization, is fundamental in this paper.

**Lemma 3.2.9**  $\mathcal{SN}_\beta \in \mathcal{SAT}_\beta$ .

PROOF. We check the clauses  $(\mathcal{SAT1})$  and  $(\mathcal{SAT2}_\beta)$ .

$(\mathcal{SAT1})$  By induction on  $E[\ ] \in \mathcal{SN}_\beta$ , using property (11).

$(\mathcal{SAT2}_\beta)$  Assume that  $E[t[u/x]], u \in \mathcal{SN}_\beta$ . We must show that  $E[(\lambda x.t)u] \in \mathcal{SN}_\beta$ , hence that for all  $v$ , if  $E[(\lambda x.t)u] \rightarrow_\beta v$  then  $v \in \mathcal{SN}_\beta$ . Note that  $E[\ ], t \in \mathcal{SN}_\beta$  since  $E[t[u/x]] \in \mathcal{SN}$ . We reason by induction on tuples  $(E[\ ], t, u)$  ordered by the product extension of  $\rightarrow_\beta$ .

Let  $v$  such that  $E[(\lambda x.t)u] \rightarrow_\beta v$ . By property (12), there are two cases.

- $v = E'[(\lambda x.t)u]$  with  $E[\ ] \rightarrow_\beta E'[\ ]$ . In this case, we conclude by induction hypothesis. Note that  $E'[t[u/x]] \in \mathcal{SN}_\beta$  since  $E[\ ] \rightarrow_\beta E'[\ ]$ .
- $v = E[s]$  with  $(\lambda x.t)u \rightarrow_\beta s$ . By Lem 3.2.8, there are two subcases.
  - $s = (\lambda x.t')u'$  with  $(t, u) \rightarrow_\beta (t', u')$ . In this case, we conclude by induction hypothesis. Note that  $E[t'[u'/x]] \in \mathcal{SN}_\beta$  since  $(t, u) \rightarrow_\beta (t', u')$ .
  - $s = t[u/x]$ . Since  $E[t[u/x]] \in \mathcal{SN}_\beta$  by assumption. □

We now show that  $A \Rightarrow B \in \mathcal{SAT}_\beta$  for all  $A, B \in \mathcal{SAT}_\beta$ . Since  $\llbracket A \Rightarrow B \rrbracket = \llbracket A \rrbracket \Rightarrow \llbracket B \rrbracket$ , this implies the adequacy of  $\llbracket - \rrbracket$  and that  $\mathcal{X} \subseteq \llbracket T \rrbracket \subseteq \mathcal{SN}_\beta$  for all  $T \in \mathcal{T}_\Rightarrow(\mathcal{B})$ . It follows that  $\llbracket - \rrbracket$  is a type interpretation in the sense of Def. 2.2.1.

**Proposition 3.2.10** *If  $A, B \in \mathcal{SAT}_\beta$  then  $A \Rightarrow B \in \mathcal{SAT}_\beta$ .*

PROOF. We first show that  $A \Rightarrow B \subseteq \mathcal{SN}_\beta$ . If  $t \in A \Rightarrow B$ , since  $A \in \mathcal{SAT}_\beta$  we have  $tx \in B$  for all  $x \in \mathcal{X}$ , hence  $tx \in \mathcal{SN}_\beta$  because  $B \in \mathcal{SAT}_\beta$ . It follows that  $t \in \mathcal{SN}_\beta$ .

We now check the clauses  $(\mathcal{SAT1})$  and  $(\mathcal{SAT2}_\beta)$ .

$(\mathcal{SAT1})$  If  $E[\ ] \in \mathcal{SN}$  and  $u \in A$ , since  $A \in \mathcal{SAT}_\beta$  we have  $u \in \mathcal{SN}$ , hence  $E[x]u \in B$  for all  $x \in \mathcal{X}$  since  $B \in \mathcal{SAT}_\beta$ . It follows that  $E[x] \in A \Rightarrow B$  for all  $x \in \mathcal{X}$ .

$(\mathcal{SAT2}_\beta)$  If  $E[t[u/x]] \in A \Rightarrow B$  with  $u \in \mathcal{SN}$ , then for all  $v \in A$  we have  $E[t[u/x]]x \in B$ , hence  $E[(\lambda x.t)u]v \in B$  because  $B \in \mathcal{SAT}_\beta$ . It follows that  $E[(\lambda x.t)u] \in A \Rightarrow B$ . □

It follows that  $\llbracket - \rrbracket : \mathcal{T}_\Rightarrow(\mathcal{B}) \mapsto \mathcal{SAT}_\beta$ . We conclude the section by showing that  $\llbracket - \rrbracket$  is indeed an adequate interpretation. As usual, this is proved by induction on typing derivations.

**Lemma 3.2.11 (Adequacy)** *If  $\Gamma \vdash_{\Rightarrow} t : T$  and  $\sigma \models \Gamma$  then  $t\sigma \in \llbracket T \rrbracket$ .*

PROOF. By induction on  $\Gamma \vdash_{\Rightarrow} t : T$ .

(AX)

$$\overline{\Gamma, x : T \vdash x : T}$$

We have  $\sigma(x) \in \llbracket T \rrbracket$  by assumption.

( $\Rightarrow$ I)

$$\frac{\Gamma, x : U \vdash t : T}{\Gamma \vdash \lambda x. t : U \Rightarrow T}$$

Let  $\sigma \models \Gamma$ . We have to show that  $(\lambda x. t)\sigma \in \llbracket U \rrbracket \Rightarrow \llbracket T \rrbracket$ .

First, we can assume that  $x \notin \text{FV}(\sigma) \cup \text{Dom}(\sigma)$ . Hence we have  $(\lambda x. t)\sigma = \lambda x. (t\sigma)$ . Now, let  $u \in \llbracket U \rrbracket$ . By induction hypothesis, we have  $t(\sigma[u/x]) \in \llbracket T \rrbracket$ . Moreover, we have  $t(\sigma[u/x]) = (t\sigma)[u/x]$  because  $x \notin \text{FV}(\sigma)$ . By (SAT<sub>2 $\beta$</sub> ), we have  $(\lambda x. t\sigma)u \in \llbracket T \rrbracket$  because  $u \in \llbracket U \rrbracket \subseteq \mathcal{SN}_{\beta}$  and  $t\sigma[u/x] \in \llbracket T \rrbracket$ . It follows that  $\lambda x. (t\sigma) \in \llbracket U \rrbracket \Rightarrow \llbracket T \rrbracket$ .

( $\Rightarrow$ E)

$$\frac{\Gamma \vdash t : U \Rightarrow T \quad \Gamma \vdash u : U}{\Gamma \vdash tu : T}$$

By definition of  $_ \Rightarrow _$ .

□

### 3.3 Lambda-Calculus with Products

In this section, we sketch the extension to the  $\lambda$ -calculus with products of the reasoning of Sect. 3.2. As for  $_ \Rightarrow _$ , we use a definition of the product space  $_ \times _$  based on eliminations. It is defined as

$$A \times B \quad =_{\text{def}} \quad \{t \mid \pi_1 t \in A \quad \wedge \quad \pi_2 t \in B\}.$$

The interpretation  $\llbracket _ \rrbracket$  is then extended to products as follows:

$$\begin{aligned} \llbracket B \rrbracket &=_{\text{def}} \mathcal{SN}_{\beta\pi} && \text{if } B \in \mathcal{B}, \\ \llbracket U \Rightarrow T \rrbracket &=_{\text{def}} \llbracket U \rrbracket \Rightarrow \llbracket T \rrbracket, \\ \llbracket U \times T \rrbracket &=_{\text{def}} \llbracket U \rrbracket \times \llbracket T \rrbracket. \end{aligned}$$

The elimination contexts are the straightforward extension of those defined in Def. 3.2.5. The set  $\mathcal{E}_{\Rightarrow \times}$  is generated by the following grammar:

$$E[ ] \in \mathcal{E}_{\Rightarrow \times} \quad ::= \quad [ ] \quad | \quad E[ ] t \quad | \quad \pi_1 E[ ] \quad | \quad \pi_2 E[ ],$$

where  $t \in \Lambda(\Sigma)$ . As for the pure  $\lambda$ -calculus, property (5) leads us to saturated sets.

**Definition 3.3.1 (Saturated Sets)** *The set SAT <sub>$\beta\pi$</sub>  of  $\beta\pi$ -saturated sets is the set of all  $S \subseteq \mathcal{SN}_{\beta\pi}$  such that*

- (SAT<sub>1</sub>) if  $E[ ] \in \mathcal{SN}_{\beta\pi}$  and  $x \in \mathcal{X}$  then  $E[x] \in S$ ,
- (SAT<sub>2 $\beta$</sub> ) if  $E[t[u/x]] \in S$  and  $u \in \mathcal{SN}_{\beta\pi}$  then  $E[(\lambda x. t)u] \in S$ ,
- (SAT<sub>2 $\pi_1$</sub> ) if  $E[t_1] \in S$  and  $t_2 \in \mathcal{SN}_{\beta\pi}$  then  $E[\pi_1 \langle t_1, t_2 \rangle] \in S$ ,
- (SAT<sub>2 $\pi_2$</sub> ) if  $E[t_2] \in S$  and  $t_1 \in \mathcal{SN}_{\beta\pi}$  then  $E[\pi_2 \langle t_1, t_2 \rangle] \in S$ .

As for the pure  $\lambda$ -calculus, it is important to check that  $\mathcal{SN}_{\beta\pi} \in \mathcal{SAT}_{\beta\pi}$ . We rely on non-interaction properties extending (11) and (12):

$$E[x] \rightarrow_{\beta\pi} v \implies (v = E'[x] \text{ with } E[\ ] \rightarrow_{\beta\pi} E'[\ ])$$
 (13)

$$\begin{aligned} E[(\lambda x.t)u] \rightarrow_{\beta\pi} v &\implies (v = E'[s] \text{ with } (E[\ ], (\lambda x.t)u) \rightarrow_{\beta\pi} (E'[\ ], s)) \\ E[\pi_i\langle t_1, t_2 \rangle] \rightarrow_{\beta\pi} v &\implies (v = E'[s] \text{ with } (E[\ ], \pi_i\langle t_1, t_2 \rangle) \rightarrow_{\beta\pi} (E'[\ ], s)) \end{aligned}$$
 (14)

Weak standardization is also a direct extension of Lem 3.2.8.

$$(\lambda x.t)u \rightarrow_{\beta\pi} v \implies (v = t[u/x] \text{ or } v = (\lambda x.t')u' \text{ with } (t, u) \rightarrow_{\beta\pi} (t', u'))$$
 (15)

$$\pi_i\langle t_1, t_2 \rangle \rightarrow_{\beta\pi} v \implies (v = t_i \text{ or } v = \pi_i\langle t'_1, t'_2 \rangle \text{ with } (t_1, t_2) \rightarrow_{\beta\pi} (t'_1, t'_2))$$
 (16)

Reasoning as in Lem. 3.2.9, we obtain the non-emptiness of  $\mathcal{SAT}_{\beta\pi}$ .

**Lemma 3.3.2**  $\mathcal{SN}_{\beta\pi} \in \mathcal{SAT}_{\beta\pi}$ .

PROOF. We only check the clauses  $(\mathcal{SAT}_{2\pi 1})$  and  $(\mathcal{SAT}_{2\pi 2})$ . We have to show that for all  $i \in \{1, 2\}$ ,

$$(t_{3-i} \in \mathcal{SN}_{\beta\pi} \wedge E[t_i] \in \mathcal{SN}_{\beta\pi}) \implies E[\pi_i\langle t_1, t_2 \rangle] \in \mathcal{SN}_{\beta\pi}.$$

We show that for all  $v$ ,  $E[\pi_i\langle t_1, t_2 \rangle] \rightarrow_{\beta\pi} v$  implies  $v \in \mathcal{SN}_{\beta\pi}$ . We reason by induction on  $(E[\ ], t_1, t_2)$  ordered by the product extension of  $\rightarrow_{\beta\pi}$ .

Let  $v$  such that  $E[\pi_i\langle t_1, t_2 \rangle] \rightarrow_{\beta\pi} v$ . By property (14), there are two cases.

- $v = E'[\pi_i\langle t_1, t_2 \rangle]$  with  $E[\ ] \rightarrow_{\beta\pi} E'[\ ]$ . In this case, we conclude by induction hypothesis. Note that  $E'[t_i] \in \mathcal{SN}_{\beta\pi}$  since  $E[\ ] \rightarrow_{\beta\pi} E'[\ ]$ .
- $v = E[s]$  with  $\pi_i\langle t_1, t_2 \rangle \rightarrow_{\beta\pi} s$ . By property (16) there are two subcases.
  - $s = \pi_i\langle t'_1, t'_2 \rangle$  with  $(t_1, t_2) \rightarrow_{\beta\pi} (t'_1, t'_2)$ . In this case, we conclude by induction hypothesis. Note that  $E[t'_i] \in \mathcal{SN}_{\beta\pi}$  since  $(t_1, t_2) \rightarrow_{\beta\pi} (t'_1, t'_2)$ .
  - $s = t_i$ . Since  $E[t_i] \in \mathcal{SN}_{\beta\pi}$  by assumption.  $\square$

Finally, we have  $A_1 \times A_2, A_2 \Rightarrow A_1 \in \mathcal{SAT}_{\beta\pi}$  for all  $A_1, A_2 \in \mathcal{SAT}_{\beta\pi}$ . The proof is similar to that of Prop. 3.2.10.

**Proposition 3.3.3** *If  $A_1, A_2 \in \mathcal{SAT}_{\beta\pi}$  then*

$$A_2 \Rightarrow A_1 \in \mathcal{SAT}_{\beta\pi}, \quad (\Rightarrow)$$

$$A_1 \times A_2 \in \mathcal{SAT}_{\beta\pi}. \quad (\times)$$

PROOF. The fact that  $A_2 \Rightarrow A_1 \in \mathcal{SN}_{\beta\pi}$  can be shown exactly as in Prop. 3.2.10. Moreover, if  $t \in A_1 \times A_2$ , then  $\pi_i t \in A_i \subseteq \mathcal{SN}_{\beta\pi}$ , hence  $t \in \mathcal{SN}_{\beta\pi}$ . In the case of  $(\Rightarrow)$ , the clause  $(\mathcal{SAT}_1)$  is dealt with as in Prop. 3.2.10. The case of  $(\times)$  is similar.

$(\mathcal{SAT}_1)$  If  $E[\ ] \in \mathcal{SN}$  and  $x \in \mathcal{X}$ , then  $\pi_i E[x] \in A_i$  because  $A_i \in \mathcal{SAT}_{\beta\pi}$  and  $\pi_i E[\ ] \in \mathcal{E}_{\Rightarrow \times}$ . It follows that  $E[x] \in A_1 \times A_2$  for all  $x \in \mathcal{X}$ .

We now consider the clauses  $(\mathcal{SAT}_{2\beta})$  and  $(\mathcal{SAT}_{2\pi i})$ . The satisfaction of  $(\mathcal{SAT}_{2\beta})$  in the case of  $(\Rightarrow)$  can be shown as in Prop. 3.2.10. However, it is interesting to see that  $(\Rightarrow)$  and  $(\times)$  can be dealt with in a uniform way. Let  $i \in \{1, 2\}$  and  $t \mapsto_{\beta\pi} u$  with

$$t = (\lambda x.t_1)t_2 \quad \text{and} \quad u = t_1[t_2/x], \quad (\mathcal{SAT}_{2\beta})$$

$$t = \pi_i\langle t_1, t_2 \rangle \quad \text{and} \quad u = t_i. \quad (\mathcal{SAT}_{2\pi i})$$

Assume that  $t_2 \in \mathcal{SN}_{\beta\pi}$  in the case of  $(\mathcal{SAT}_{2\beta})$  and that  $t_{3-i} \in \mathcal{SN}_{\beta\pi}$  in the case of  $(\mathcal{SAT}_{2\pi i})$ .



( $\Rightarrow$ ) We have  $E[u] \in A_2 \Rightarrow A_1$  and we must show that  $E[t] \in A_2 \Rightarrow A_1$ . For all  $v \in A_2$ , since  $E[\ ]v \in \mathcal{E}_{\Rightarrow \times}$  and  $E[u]v \in A_1$ , we have  $E[t]v \in A_1$  because  $A_1 \in \mathcal{SAT}_{\beta\pi}$ . It follows that  $E[t] \in A_2 \Rightarrow A_1$ .

( $\times$ ) We have  $E[u] \in A_1 \times A_2$  and we must show that  $E[t] \in A_1 \times A_2$ . For all  $j \in \{1, 2\}$ , since  $\pi_j E[\ ] \in \mathcal{E}_{\Rightarrow \times}$  and  $\pi_j E[u] \in A_j$ , we have  $\pi_j E[t] \in A_j$  because  $A_j \in \mathcal{SAT}_{\beta\pi}$ . It follows that  $E[t] \in A_1 \times A_2$ .  $\square$

It follows that  $\llbracket \_ \rrbracket : \mathcal{T}_{\Rightarrow \times}(\mathcal{B}) \mapsto \mathcal{SAT}_{\beta\pi}$ . We now sketch the adequacy proof of  $\llbracket \_ \rrbracket$ .

**Lemma 3.3.4 (Adequacy)** *If  $\Gamma \vdash_{\Rightarrow \times} t : T$  and  $\sigma \models \Gamma$  then  $t\sigma \in \llbracket T \rrbracket$ .*

PROOF. By induction on  $\Gamma \vdash_{\Rightarrow \times} t : T$ . We only detail the cases which differ from Lem. 3.2.11.

( $\times I$ )

$$\frac{\Gamma \vdash t_1 : T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash \langle t_1, t_2 \rangle : T_1 \times T_2}$$

Let  $\sigma \models \Gamma$ . We have to show that  $\langle t_1, t_2 \rangle \sigma = \langle t_1 \sigma, t_2 \sigma \rangle \in \llbracket T_1 \rrbracket \times \llbracket T_2 \rrbracket$ . By induction hypothesis, we have  $t_1 \sigma \in \llbracket T_1 \rrbracket$  and  $t_2 \sigma \in \llbracket T_2 \rrbracket$ . It follows that for all  $i \in \{1, 2\}$  we have  $\pi_i \langle t_1 \sigma, t_2 \sigma \rangle \in \llbracket T_i \rrbracket$  by  $(\mathcal{SAT}_{2\pi i})$ , hence  $\langle t_1 \sigma, t_2 \sigma \rangle \in \llbracket T_1 \rrbracket \times \llbracket T_2 \rrbracket$ .

( $\times E$ )

$$\frac{\Gamma \vdash t : T_1 \times T_2}{\Gamma \vdash \pi_i t : T_i} \quad (i \in \{1, 2\})$$

By definition of  $\_ \times \_$ .  $\square$

### 3.4 The Closure Operator of Saturated Sets

In this section, we check that  $\mathcal{SAT}_{\beta}$  and  $\mathcal{SAT}_{\beta\pi}$  are indeed reducibility families in the sense of Def. 2.2.7. Since we know that  $\mathcal{X} \subseteq \mathcal{S}$  and  $\mathcal{S} \subseteq \mathcal{SN}_{\beta}$  (resp.  $\mathcal{S} \subseteq \mathcal{SN}_{\beta\pi}$ ) for all  $\mathcal{S} \in \mathcal{SAT}_{\beta}$  (resp. all  $\mathcal{S} \in \mathcal{SAT}_{\beta\pi}$ ), it remains to show that  $\mathcal{SAT}_{\beta}$  (resp.  $\mathcal{SAT}_{\beta\pi}$ ) is given by a closure operator. We focus on the closure operator of  $\mathcal{SAT}_{\beta\pi}$ .

**Definition 3.4.1** *We define the functions  $\mathcal{SAT}_i : \mathcal{P}(\mathcal{SN}_{\beta\pi}) \mapsto \mathcal{P}(\mathcal{SN}_{\beta\pi})$  by induction on  $i \in \mathbb{N}$  as follows*

$$\begin{aligned} \mathcal{SAT}_0(X) &=_{\text{def}} X \cup \{E[x] \mid E[\ ] \in \mathcal{E}_{\Rightarrow \times} \cap \mathcal{SN}_{\beta\pi} \wedge x \in \mathcal{X}\}, \\ \mathcal{SAT}_{i+1}(X) &=_{\text{def}} \mathcal{SAT}_i(X) \cup \{E[(\lambda x.t)u] \mid E[t[u/x]] \in \mathcal{SAT}_i(X) \wedge u \in \mathcal{SN}_{\beta\pi}\} \\ &\quad \cup \{E[\pi_1 \langle t_1, t_2 \rangle] \mid E[t_1] \in \mathcal{SAT}_i(X) \wedge t_2 \in \mathcal{SN}_{\beta\pi}\} \\ &\quad \cup \{E[\pi_2 \langle t_1, t_2 \rangle] \mid E[t_2] \in \mathcal{SAT}_i(X) \wedge t_1 \in \mathcal{SN}_{\beta\pi}\}. \end{aligned}$$

We define the function  $\mathcal{SAT} : \mathcal{P}(\mathcal{SN}_{\beta\pi}) \mapsto \mathcal{P}(\mathcal{SN}_{\beta\pi})$  as  $\mathcal{SAT}(X) =_{\text{def}} \bigcup_{i \in \mathbb{N}} \mathcal{SAT}_i(X)$ .

We now show that  $\mathcal{SAT}(\_)$  is a closure operator defining the  $\beta\pi$ -saturated sets, in the sense of Def. 3.3.1.

**Lemma 3.4.2** *If  $X \subseteq \mathcal{SN}_{\beta\pi}$ , then  $\mathcal{SAT}(X)$  is the least  $\beta\pi$ -saturated set containing  $X$ .*

PROOF. We first show that  $\mathcal{SAT}(X) \in \mathcal{SAT}_{\beta\pi}$  for all  $X \subseteq \mathcal{SN}_{\beta\pi}$ . We begin by checking that  $\mathcal{SAT}_i(X) \subseteq \mathcal{SN}_{\beta\pi}$  for all  $i \in \mathbb{N}$ . We reason by induction on  $i$ .

**Base case ( $i = 0$ ).** Since  $X \subseteq \mathcal{SN}_\beta$ , we have  $\mathcal{SAT}_0(X) \subseteq \mathcal{SN}_{\beta\pi}$  because  $E[\ ] \in \mathcal{SN}_{\beta\pi}$  implies  $E[x] \in \mathcal{SN}_{\beta\pi}$  by non-interaction (property (13)).

**Induction case.** Let  $t \in \mathcal{SAT}_{i+1}(X)$ . If  $t \in \mathcal{SAT}_i(X)$  then  $t \in \mathcal{SN}_{\beta\pi}$  by induction hypothesis. Otherwise,  $t \mapsto_{\beta\pi} u$  and we reason as in Lem. 3.2.9 and Lem. 3.3.2.

We now check that clauses  $(\mathcal{SAT}_1)$ ,  $(\mathcal{SAT}_{2\beta})$  and  $(\mathcal{SAT}_{2\pi i})$  are satisfied by  $\mathcal{SAT}(X)$ .

$(\mathcal{SAT}_1)$ . If  $E[\ ] \in \mathcal{SN}_{\beta\pi}$ , then we have  $E[x] \in \mathcal{SAT}_0 \subseteq \mathcal{SAT}(X)$ .

$(\mathcal{SAT}_{2\beta})$  and  $(\mathcal{SAT}_{2\pi i})$ . We only detail  $(\mathcal{SAT}_{2\beta})$ . If  $E[t[u/x]] \in \mathcal{SAT}(X)$ , then there is  $i \in \mathbb{N}$  such that  $E[t[u/x]] \in \mathcal{SAT}_i(X)$ . We then deduce that  $E[(\lambda x.t)u] \in \mathcal{SAT}_{i+1}(X) \subseteq \mathcal{SAT}(X)$  if moreover  $u \in \mathcal{SN}_{\beta\pi}$ .

We now show that given  $X \subseteq \mathcal{SN}_{\beta\pi}$ , we have  $\mathcal{SAT}_i(X) \subseteq S$  for all  $S \in \mathcal{SAT}_{\beta\pi}$  such that  $X \subseteq S$ . We reason by induction on  $i$ .

**Base case ( $i = 0$ ).** Since  $S \in \mathcal{SAT}_{\beta\pi}$  and  $X \subseteq S$ .

**Induction case.** Assume that  $t \in \mathcal{SAT}_{i+1}(X)$ . If  $t \in \mathcal{SAT}_i(X)$  then  $t \in S$  by induction hypothesis. Otherwise,  $t = E[u]$  with  $u \mapsto_{\beta\pi} v$  and  $E[v] \in \mathcal{SAT}_i(X)$ . We get  $E[v] \in S$  by induction hypothesis and deduce  $E[u] \in S$  by  $(\mathcal{SAT}_{2\beta})$  and  $(\mathcal{SAT}_{2\pi i})$ .  $\square$

It follows that

$$\mathcal{SAT}_{\beta\pi} = \{\mathcal{SAT}(X) \mid X \subseteq \mathcal{SN}_{\beta\pi}\}.$$

We now show that  $\mathcal{SAT} : \mathcal{P}(\mathcal{SN}_{\beta\pi}) \mapsto \mathcal{P}(\mathcal{SN}_{\beta\pi})$  is a closure operator.

**Lemma 3.4.3**  $\mathcal{SAT} : \mathcal{P}(\mathcal{SN}_{\beta\pi}) \mapsto \mathcal{P}(\mathcal{SN}_{\beta\pi})$  is a closure operator.

PROOF.

- Idempotency. By Lem. 3.4.2.
- Extensivity. Let  $X \in \mathcal{P}(\mathcal{SN}_{\beta\pi})$ . By induction on  $i$ , we get  $X \subseteq \mathcal{SAT}_i(X)$  for all  $i \in \mathbb{N}$ , hence  $X \subseteq \mathcal{SAT}(X) = \bigcup_{i \in \mathbb{N}} \mathcal{SAT}_i(X)$ .
- Monotony. Let  $X \subseteq Y$  with  $X, Y \in \mathcal{P}(\mathcal{SN}_{\beta\pi})$ . By induction on  $i \in \mathbb{N}$ , we get  $\mathcal{SAT}_i(X) \subseteq \mathcal{SAT}_i(Y)$ . We deduce that  $\mathcal{SAT}(X)_i \subseteq \bigcup_{j \in \mathbb{N}} \mathcal{SAT}_j(Y)$  for all  $i$ , hence that  $\bigcup_{i \in \mathbb{N}} \mathcal{SAT}_i(X) \subseteq \bigcup_{j \in \mathbb{N}} \mathcal{SAT}_j(Y)$ .  $\square$

Since  $\mathcal{SAT}_{\beta\pi}$  is defined by the closure operator  $\mathcal{SAT} : \mathcal{P}(\mathcal{SN}_{\beta\pi}) \mapsto \mathcal{P}(\mathcal{SN}_{\beta\pi})$ , according to Lem. 2.2.6, it is a complete lattice whose maximal element is  $\mathcal{SN}_{\beta\pi}$  and whose g.l.b.'s are given by intersections.

### 3.5 Lambda-Calculus with Rewriting

In Sect. 3.2 and Sect. 3.3, we have sketched some basic mechanisms of reducibility for the pure  $\lambda$ -calculus and the  $\lambda$ -calculus with binary products. In the section, we consider the case of typed rewriting. Our objective is to present some general principles of the insertion of a rewrite relation in the reducibility proofs presented in Sect. 3.2 and Sect. 3.3. They will lead us somehow outside the framework of saturated sets. We build on [BJO02, Bla07].

Let  $\mathcal{R}$  be a rewrite system typed in the  $\lambda$ -calculus with products. We begin by reasoning as in Sect. 3.3. Consider an interpretation of types  $T \in \mathcal{T}_{\Rightarrow \times}(\mathcal{B})$  defined as in Sect. 3.3, but with  $\llbracket B \rrbracket =_{\text{def}} \mathcal{SN}_{\beta\pi\mathcal{R}}$ :

$$\begin{aligned} \llbracket B \rrbracket &=_{\text{def}} \mathcal{SN}_{\beta\pi\mathcal{R}} && \text{if } B \in \mathcal{B}, \\ \llbracket U \Rightarrow T \rrbracket &=_{\text{def}} \llbracket U \rrbracket \Rightarrow \llbracket T \rrbracket, \\ \llbracket T \times U \rrbracket &=_{\text{def}} \llbracket T \rrbracket \times \llbracket U \rrbracket. \end{aligned} \quad (17)$$

We use the elimination contexts of Sect. 3.3, and we consider saturated sets  $\mathcal{SAT}_{\beta\pi}$  defined as in Def. 3.3.1, but using  $\mathcal{SN}_{\beta\pi\mathcal{R}}$  instead of  $\mathcal{SN}_{\beta\pi}$ . Following the reasoning of Sect. 3.2 and Sect. 3.3, we can show that the interpretation  $\llbracket \_ \rrbracket$  defined above maps types  $T \in \mathcal{T}_{\Rightarrow \times}(\mathcal{B})$  to saturated sets  $\llbracket T \rrbracket \in \mathcal{SAT}_{\beta\pi}$ . Moreover, the non-interaction properties (11), (12) and (14), as well as the weak standardization properties (15) and (16) are still valid, using the rewrite relation  $\rightarrow_{\beta\pi\mathcal{R}}$  instead of  $\rightarrow_{\beta}$  and  $\rightarrow_{\beta\pi}$ . This implies that  $\llbracket \_ \rrbracket$  is a type interpretation in the sense of Def. 2.2.1 and ensures the adequacy of the typing rules  $(\Rightarrow I)$ ,  $(\Rightarrow E)$ ,  $(\times I)$  and  $(\times E)$ .

We now discuss the rule

$$(\text{SYMB}) \quad \frac{\Gamma \vdash t_1 : T_1 \quad \dots \quad \Gamma \vdash t_n : T_n}{\Gamma \vdash f(t_1, \dots, t_n) : T} \quad (T_1, \dots, T_n, T) \in \tau(f)$$

The interpretation  $\llbracket \_ \rrbracket$  is adequate for this rule if we have  $f(t_1, \dots, t_n) \in \llbracket T \rrbracket$  as soon as  $t_i \in \llbracket T_i \rrbracket$  for all  $i \in \{1, \dots, n\}$ . We now take a look at the closure properties of  $\llbracket \_ \rrbracket$  that may allow to take rewriting into account. As for the  $\lambda$ -calculus with products, we can use a non-interaction property similar to (12) and (14): for all  $E[\_] \in \mathcal{E}_{\Rightarrow \times}$ ,

$$E[f(t_1, \dots, t_n)] \rightarrow_{\beta\pi\mathcal{R}} v \implies (v = E'[s] \text{ with } (E[\_], f(t_1, \dots, t_n)) \rightarrow_{\beta\pi\mathcal{R}} (E'[\_], s)) . \quad (18)$$

But rewrite systems do not satisfy in general the weak standardization lemma. Therefore, as shown by the following example, there are rewrite systems such that given  $E[\_] \in \mathcal{SN}_{\beta\mathcal{R}}$ , in order to get  $E[f(t_1, \dots, t_n)] \in \mathcal{SN}_{\beta\mathcal{R}}$  we need  $v \in \mathcal{SN}_{\beta\pi\mathcal{R}}$  for all  $v$  such that  $E[f(t_1, \dots, t_n)] \rightarrow_{\beta\mathcal{R}} v$ .

**Example 3.5.1** *Consider the confluent system*

$$p \mapsto_{\mathcal{R}} \lambda x. \lambda y. \lambda z. g(xy) \quad p \mapsto_{\mathcal{R}} \lambda x. \lambda y. \lambda z. g(xz) \quad g(x) \mapsto_{\mathcal{R}} a$$

whose symbols are typed as follows:

$$\frac{}{\Gamma \vdash p : (B \Rightarrow B) \Rightarrow B \Rightarrow B \Rightarrow B} \quad \frac{\Gamma \vdash t : B}{\Gamma \vdash g(t) : B} \quad \frac{}{\Gamma \vdash a : B}$$

We have  $p \in \mathcal{SN}_{\beta\mathcal{R}}$ , but there are untyped elimination contexts which separate the terms

$$\lambda x. \lambda y. \lambda z. g(xy) \quad \text{and} \quad \lambda x. \lambda y. \lambda z. g(xz)$$

with respect to strong normalization. For instance, we have

$$\begin{aligned} (\lambda x. \lambda y. \lambda z. g(xy)) \delta a \delta &\in \mathcal{SN}_{\beta\mathcal{R}} & (\lambda x. \lambda y. \lambda z. g(xz)) \delta a \delta &\notin \mathcal{SN}_{\beta\mathcal{R}} \\ (\lambda x. \lambda y. \lambda z. g(xy)) \delta \delta a &\notin \mathcal{SN}_{\beta\mathcal{R}} & (\lambda x. \lambda y. \lambda z. g(xz)) \delta \delta a &\in \mathcal{SN}_{\beta\mathcal{R}} \end{aligned}$$

where  $\delta =_{\text{def}} \lambda x. x x$ , hence  $\delta \delta \notin \mathcal{SN}_{\beta}$ . This example is related to the problem of stability by union, and will be explained using a weak observational preorder in Sect. 6.4.

Hence, for a given rewrite system  $\mathcal{R}$ , there may not be any notion of weak-head expansion which preserves strong normalization. Therefore, we need a stronger clause than  $(\mathcal{SAT}2_\beta)$  and  $(\mathcal{SAT}2_{\pi i})$ . Example 3.5.1 shows that in general we can not do better than

$$(\forall v. \ E[f(t_1, \dots, t_n)] \rightarrow_{\beta\pi\mathcal{R}} v \implies v \in \llbracket T \rrbracket) \implies E[f(t_1, \dots, t_n)] \in \llbracket T \rrbracket. \quad (19)$$

Property (19) holds for the interpretation  $\llbracket \cdot \rrbracket$ . In order to prove it, we need to show the stability by reduction of  $\llbracket T \rrbracket$ .

**Proposition 3.5.2** *If  $t \in \llbracket T \rrbracket$  and  $t \rightarrow_{\beta\pi\mathcal{R}} u$  then  $u \in \llbracket T \rrbracket$ .*

PROOF. By induction on  $T$ . Let  $t \in \llbracket T \rrbracket$  and  $u$  such that  $t \rightarrow_{\beta\pi\mathcal{R}} u$ .

$T = B \in \mathcal{B}$ . In this case, we have  $\llbracket T \rrbracket = \mathcal{SN}_{\beta\pi\mathcal{R}}$ , which is stable by reduction.

$T = T_2 \Rightarrow T_1$  and  $T = T_1 \times T_2$ . We only detail  $T_2 \Rightarrow T_1$  since  $T_1 \times T_2$  is similar and simpler.

Let  $v \in \llbracket T_2 \rrbracket$ . Since  $t \in \llbracket T_2 \rrbracket \Rightarrow \llbracket T_1 \rrbracket$ , we have  $tv \in \llbracket T_1 \rrbracket$ , hence  $uv \in \llbracket T_1 \rrbracket$  by induction hypothesis. It follows that  $u \in \llbracket T_2 \rrbracket \times \llbracket T_1 \rrbracket$ .  $\square$

Note that stability by reduction is not a property specified by saturated sets, even for the pure  $\lambda$ -calculus (see for instance [Wer94]).

**Example 3.5.3** *Consider the terms  $t =_{\text{def}} \lambda x. (\lambda y. y)x$  and  $u =_{\text{def}} \lambda x. x$ . We have  $t \rightarrow_\beta u$ , but  $u \notin \mathcal{SAT}(\{t\})$  and  $t \notin \mathcal{SAT}(\{u\})$ , where  $\mathcal{SAT}(\cdot)$  is the closure operator of  $\mathcal{SAT}_{\beta\pi}$  defined in Def. 3.4.1.*

PROOF. It is sufficient to note that for all  $X \subseteq \mathcal{SN}_\beta$ , all terms of the form  $\lambda x. v$  which belong to  $\mathcal{SAT}(X)$  already belong to  $X$ .  $\square$

We now show property (19).

**Proposition 3.5.4** *For all  $T \in \mathcal{T}_{\Rightarrow \times}(\mathcal{B})$ ,*

$$(\forall v. \ E[f(t_1, \dots, t_n)] \rightarrow_{\beta\pi\mathcal{R}} v \implies v \in \llbracket T \rrbracket) \implies E[f(t_1, \dots, t_n)] \in \llbracket T \rrbracket.$$

PROOF. By induction on  $T$ .

$T = B \in \mathcal{B}$ . In this case  $\llbracket T \rrbracket = \mathcal{SN}_{\beta\pi\mathcal{R}}$  and we conclude by definition of  $\mathcal{SN}_{\beta\pi\mathcal{R}}$ .

$T = T_2 \Rightarrow T_1$  and  $T = T_1 \times T_2$ . We only detail  $T_2 \Rightarrow T_1$  since  $T_1 \times T_2$  is similar and simpler. Let  $u \in \llbracket T_2 \rrbracket$ . We reason by induction on  $u \in \mathcal{SN}_{\beta\pi\mathcal{R}}$ . By non-interaction (property (18)), if  $E[f(\vec{t})]u \rightarrow_{\beta\pi\mathcal{R}} v$ , then  $v = E'[t']u'$  with  $(E[\ ], f(\vec{t}), u) \rightarrow_{\beta\pi\mathcal{R}} (E'[\ ], t', u')$ . There are two cases.

- $E[f(\vec{t})] \rightarrow_{\beta\pi\mathcal{R}} E'[t']$ . We have  $E'[t'] \in \llbracket T_2 \rrbracket \Rightarrow \llbracket T_1 \rrbracket$  by assumption, hence  $v \in \llbracket T_1 \rrbracket$ .
- $u \rightarrow_{\beta\pi\mathcal{R}} u'$ . We have  $u' \in \llbracket T_2 \rrbracket$  since  $\llbracket T_2 \rrbracket$  is stable by reduction (Prop. 3.5.2), and we conclude by induction hypothesis.  $\square$

Let us apply Prop. 3.5.4 and Prop. 3.5.2 to show the correctness of the rule (SYMB). Assume that

$$\frac{\Gamma \vdash t_1 : T_1 \quad \dots \quad \Gamma \vdash t_n : T_n}{\Gamma \vdash f(t_1, \dots, t_n) : T} (T_1, \dots, T_n, T) \in \tau(f)$$

By Prop. 3.5.4, we have  $f(t_1, \dots, t_n) \in \llbracket T \rrbracket$  if  $v \in \llbracket T \rrbracket$  for all  $v$  such that  $f(t_1, \dots, t_n) \rightarrow_{\beta\pi\mathcal{R}} v$ . Let  $t_i \in \llbracket T_i \rrbracket$  for all  $i \in \{1, \dots, n\}$ . We reason by induction on the tuple  $(t_1, \dots, t_n)$  ordered by the product extension of  $\rightarrow_{\beta\pi\mathcal{R}}$ . Let  $v$  such that  $f(t_1, \dots, t_n) \rightarrow_{\beta\pi\mathcal{R}} v$ . There are two cases:

- $v = f(t'_1, \dots, t'_n)$  with  $(t_1, \dots, t_n) \rightarrow_{\beta\pi\mathcal{R}} (t'_1, \dots, t'_n)$ . By Prop. 3.5.2 we have  $t'_i \in \llbracket T_i \rrbracket$  for all  $i \in \{1, \dots, n\}$ , and we conclude by induction hypothesis.
- There is a rule  $f(l_1, \dots, l_n) \mapsto_{\mathcal{R}} r$  and a substitution  $\sigma$  such that  $v = r\sigma$  and  $t_i = l_i\sigma$  for all  $i \in \{1, \dots, n\}$ . In this case, we could conclude if we knew that  $r\sigma \in \llbracket T \rrbracket$  for all  $\sigma$  such that  $l_i\sigma \in \llbracket T_i \rrbracket$  for all  $i \in \{1, \dots, n\}$ .

The sufficient condition of the second case is the *computability of rewrite rules*. This is a well-known condition [Bla07], which is at the basis of some termination criteria for the combination of rewriting with typed  $\lambda$ -calculi [BR06, BJO02]. As shown above, this implies the adequacy of the interpretation  $\llbracket \_ \rrbracket$ .

**Lemma 3.5.5 (Computability of Rewrite Rules)** *Let  $\mathcal{R}$  be a rewrite system typed in  $\vdash_{\Rightarrow \times \tau}$ . Then,  $\llbracket \_ \rrbracket$  is adequate for  $\vdash_{\Rightarrow \times \tau}$  if for all rule  $f(l_1, \dots, l_n) \mapsto_{\mathcal{R}} r$  with*

$$\frac{\Gamma \vdash t_1 : T_1 \quad \dots \quad \Gamma \vdash t_n : T_n}{\Gamma \vdash f(t_1, \dots, t_n) : T} \quad (T_1, \dots, T_n, T) \in \tau(f)$$

and all substitution  $\sigma$  we have

$$(l_1\sigma \in \llbracket T_1 \rrbracket \ \wedge \ \dots \ \wedge \ l_n\sigma \in \llbracket T_n \rrbracket) \implies r\sigma \in \llbracket T \rrbracket .$$

To summarize,  $\llbracket \_ \rrbracket$  maps types to saturated sets  $S \in \mathcal{SAT}_{\beta\pi\mathcal{R}}$  such that  $S \subseteq \mathcal{SN}_{\beta\pi\mathcal{R}}$  and

- (SAT0) if  $t \in S$  and  $t \rightarrow_{\beta\pi\mathcal{R}} u$  then  $u \in S$ ,
- (SAT1) if  $E[\_] \in \mathcal{SN}_{\beta\pi\mathcal{R}}$  and  $x \in \mathcal{X}$  then  $E[x] \in S$ ,
- (SAT2 $_{\beta}$ ) if  $E[t[u/x]] \in S$  and  $u \in \mathcal{SN}_{\beta\pi\mathcal{R}}$  then  $E[(\lambda x.t)u] \in S$ ,
- (SAT2 $_{\pi i}$ ) if  $t_{3-i} \in \mathcal{SN}_{\beta\pi\mathcal{R}}$  and  $E[t_i] \in S$  then  $E[\pi_i(t_1, t_2)] \in S$ ,
- (SAT2 $_{\mathcal{R}}$ ) if  $\forall v. E[f(t_1, \dots, t_n)] \rightarrow_{\beta\pi\mathcal{R}} v \implies v \in S$  then  $E[f(t_1, \dots, t_n)] \in S$ .

As seen above, these saturated sets work well, but their formulation is rather ad-hoc and lacks uniformity.

## 4 Neutral Terms and Reducibility Candidates

In this section we present our notion of reducibility candidates. The originality of our approach is to define neutral terms generically from a non-interaction property between terms and some contexts called elimination contexts. Thus, we can formulate a notion of reducibility candidates in a very simple framework, which only assumes a rewrite relation and a set of contexts, required to satisfy some simple properties. Our generalization comes from this principle of definition for neutral terms, while our reducibility candidates use the usual clauses [GLT89, Gal89].

Let us present the main ideas. In contrast with saturated sets, the closure properties of reducibility candidates can be formulated in a uniform way. They are based on clauses similar to (SAT0) and (SAT2 $_{\mathcal{R}}$ ), used in Sect. 3.5 above to handle rewriting in saturated sets. In particular, a reducibility candidate for rewriting  $C \subseteq \mathcal{SN}_{\beta\pi\mathcal{R}}$  satisfies

$$(t \in C \ \wedge \ t \rightarrow_{\beta\pi\mathcal{R}} u) \implies u \in C , \tag{20}$$

$$(\forall v. E[f(t_1, \dots, t_n)] \rightarrow_{\beta\pi\mathcal{R}} v \implies v \in C) \implies E[f(t_1, \dots, t_n)] \in C . \tag{21}$$

The uniformity of reducibility candidates comes from the possibility to formulate a clause, called the *neutral term property*, which implies (21) as well as (SAT1), (SAT2<sub>β</sub>) and (SAT2<sub>π</sub>). This is due to *neutral terms*, that enjoy non-interaction properties such as (11), (12) and (18), with some contexts called *elimination contexts*.

The idea developed in this section is to start from the notion of non-interaction to define reducibility candidates generically. The key point is that non-interaction can be formulated in a very simple framework, assuming only a rewrite relation  $\rightarrow_R$  on  $\Lambda(\Sigma)$  and a set  $\mathcal{E}$  of contexts called elimination contexts. More precisely, a term  $t \in \Lambda(\Sigma)$  is called *neutral* if it interacts with no contexts  $E[\ ] \in \mathcal{E}$ :

$$E[t] \rightarrow_R v \quad \text{implies} \quad (v = E'[t'] \quad \text{with} \quad (E[\ ], t) \rightarrow_R (E'[\ ], t')) . \quad (22)$$

This notion of neutral term is sufficient to formulate the *neutral term property*: if  $t$  is neutral, then any reducibility candidate  $C$  is required to satisfy

$$(\forall v. \quad t \rightarrow_R v \implies v \in C) \implies t \in C .$$

Girard's reducibility candidates rely on the fact that since neutral terms do not interact with elimination contexts, their properties w.r.t. reducibility can be shown by induction on  $\mathcal{SN}_R$ . In our framework, this means that if  $C$  is a reducibility candidate, then for all neutral term  $t$  and all elimination context  $E[\ ]$ , we have

$$(\forall u. \quad t \rightarrow_R u \implies E[u] \in C) \implies E[t] \in C . \quad (23)$$

This property is central in reducibility, it holds for saturated sets  $\mathcal{SAT}_{\beta\pi}$  and for biorthogonals. In our framework, it follows from some simple assumptions on  $\rightarrow_R$  and  $\mathcal{E}$  which define when  $\mathcal{E}$  is a set of *elimination contexts* for  $\rightarrow_R$ . We show it in Lem. 4.2.5.

We first give a general notion of neutral term in Sect. 4.1, which lead to a general notion of reducibility candidates presented in Sect. 4.2. We then study more precisely reducibility candidates. Their closure operator is defined in Sect. 4.3. From the material of Sect. 2.2 we deduce that reducibility candidates form a complete lattice, whose least element is studied in Sect. 4.4. Finally, we explore their order-theoretic structure in Sect. 4.5.

The ideas developed here were sketched in [Rib08] and Sect. 4.5 generalizes results of [Rib07b].

## 4.1 Neutral Terms

Our starting point is the fact that some terms do not interact with elimination contexts. We have seen this in Sect. 3 with properties (11), (12), (14) and (18). We recall them here:

$$\begin{aligned} E[x] \rightarrow_{\beta\pi R} v &\implies (v = E'[x] \quad \text{with} \quad E[\ ] \rightarrow_{\beta\pi R} E'[\ ]) & (24) \\ E[(\lambda x.t)u] \rightarrow_{\beta\pi R} v &\implies (v = E'[t'] \quad \text{with} \quad (E[\ ], (\lambda x.t)u) \rightarrow_{\beta\pi R} (E'[\ ], t')) \\ E[\pi_i \langle t_1, t_2 \rangle] \rightarrow_{\beta\pi R} v &\implies (v = E'[t'] \quad \text{with} \quad (E[\ ], \pi_i \langle t_1, t_2 \rangle) \rightarrow_{\beta\pi R} (E'[\ ], t')) \\ E[f(t_1, \dots, t_n)] \rightarrow_{\beta\pi R} v &\implies (v = E'[t'] \quad \text{with} \quad (E[\ ], f(t_1, \dots, t_n)) \rightarrow_{\beta\pi R} (E'[\ ], t')) . \end{aligned}$$

Terms which do not interact with elimination contexts are called *neutral*. For instance, if the contexts  $E[\ ]$  above are elimination contexts, then the following terms are neutral

$$x \qquad (\lambda x.t)u \qquad \pi_i \langle t_1, t_2 \rangle \qquad f(t_1, \dots, t_n) .$$

**Evaluation contexts.** In order to get a general notion of neutral term, we first need a general notion of elimination contexts. Elimination contexts will be defined as a subset of what we call *evaluation contexts*. The only assumption made on evaluation contexts is that they must be stable by reduction. This will be sufficient to get the instance of property (21) in the case of  $C = \mathcal{SN}_{\beta\pi\mathcal{R}}$ :

$$(\forall v. \ E[f(t_1, \dots, t_n)] \rightarrow_{\beta\pi\mathcal{R}} v \implies v \in \mathcal{SN}_{\beta\pi\mathcal{R}}) \implies E[f(t_1, \dots, t_n)] \in \mathcal{SN}_{\beta\pi\mathcal{R}}.$$

**Definition 4.1.1 (Evaluation Contexts)** Let  $[ ] \in \mathcal{X}$  be a distinguished variable and  $\rightarrow_{\mathcal{R}}$  be a rewrite relation on  $\Lambda(\Sigma)$ . A set of evaluation contexts for  $\rightarrow_{\mathcal{R}}$  is a set  $\mathcal{E}$  of terms  $E[ ]$  containing at least one occurrence of  $[ ]$ , and which is stable by reduction: if  $E[ ] \in \mathcal{E}$  and  $E[ ] \rightarrow_{\mathcal{R}} t$  then  $t = F[ ] \in \mathcal{E}$ .

If  $t \in \Lambda(\Sigma)$  and  $E[ ] \in \mathcal{E}$  then we let  $E[t] =_{\text{def}} (E[ ])[t/[ ]]$ .

**Example 4.1.2** Given a rewrite system  $\mathcal{R}$ , the contexts

$$E[ ] \in \mathcal{E}_{\Rightarrow \times} ::= [ ] \mid E[ ] t \mid \pi_1 E[ ] \mid \pi_2 E[ ]$$

defined in Sect. 3.3 are evaluation contexts for  $\rightarrow_{\beta\pi\mathcal{R}}$ .

Note that  $E[ ]$  binds no variables of  $t$  in  $E[t]$  since  $E[t] = E[ ] [t/x]$ . Allowing variable capture in evaluation contexts may lead to unexpected phenomena, see Ex. 4.1.8. The assumption that  $E[ ]$  contains at least one occurrence of  $[ ]$  ensures that  $t \in \mathcal{SN}_{\mathcal{R}}$  as soon as  $E[t] \in \mathcal{SN}_{\mathcal{R}}$ .

**Neutral terms.** We now give our formulation of the notion of neutral term, which is more general than the usual ones [GLT89, Gal89]. Neutral terms are the terms that do not interact with evaluation contexts. The generality comes from the fact that our notion of neutrality is *methodological* it relies on no particular syntactic construction. We assume given a set  $\mathcal{E}$  of evaluation contexts for  $\rightarrow_{\mathcal{R}}$ .

**Definition 4.1.3 (Neutral Terms)** A term  $t$  is neutral for  $\rightarrow_{\mathcal{R}}$  in  $\mathcal{E}$  if for all  $E[ ] \in \mathcal{E}$ ,

$$\forall v. \ E[t] \rightarrow_{\mathcal{R}} v \implies (v = E'[t'] \text{ with } (E[ ], t) \rightarrow_{\mathcal{R}} (E'[ ], t')) .$$

We denote by  $\mathcal{N}_{\mathcal{R}\mathcal{E}}$  the set of neutral terms for  $\rightarrow_{\mathcal{R}}$  in  $\mathcal{E}$ .

The set  $\mathcal{N}_{\beta}$  of neutral terms for  $\rightarrow_{\beta}$  in

$$E[ ] \in \mathcal{E}_{\Rightarrow} ::= [ ] \mid E[ ] t$$

is the set of all the terms of the form

$$E[x] \quad \text{or} \quad E[(\lambda x.t)u] \quad \text{with} \quad E[ ] \in \mathcal{E}_{\Rightarrow} .$$

The set  $\mathcal{N}_{\pi}$  of neutral terms for  $\rightarrow_{\pi}$  in

$$E[ ] \in \mathcal{E}_{\times} ::= [ ] \mid \pi_1 E[ ] \mid \pi_2 E[ ]$$

is the set of all the terms of the form

$$E[x] \quad \text{or} \quad E[\pi_i \langle t_1, t_2 \rangle] \quad \text{with} \quad E[ ] \in \mathcal{E}_{\Rightarrow} .$$

However, the shape of neutral terms for  $\rightarrow_{\beta\pi}$  in  $\mathcal{E}_{\Rightarrow \times}$  is more complex. For instance the term  $\pi_1 \lambda x.t$  is neutral.

**Values.** In fact, the terms which have the most interesting shape are the terms that are not neutral. These are the terms which interact with elimination contexts. They are therefore observable, and we call them *values*.

**Definition 4.1.4 (Values)** A value for  $\rightarrow_{\mathcal{R}}$  in  $\mathcal{E}$  is a term which is not neutral. We denote by  $\mathcal{V}_{\mathcal{R}\mathcal{E}}$  the set of values for  $\rightarrow_{\mathcal{R}}$  in  $\mathcal{E}$ .

Values are determined by the shape of evaluation contexts. The values for  $\rightarrow_{\beta\pi}$  in  $\mathcal{E}_{\Rightarrow \times}$  are exactly the terms of the form

$$\lambda x.t \quad \text{or} \quad \langle t_1, t_2 \rangle .$$

Given a rewrite system  $\mathcal{R}$ , the terms of the shape  $E[f(t_1, \dots, t_n)]$  with  $E[\ ] \in \mathcal{E}_{\Rightarrow \times}$  are all neutral for  $\rightarrow_{\beta\pi\mathcal{R}}$  in  $\mathcal{E}_{\Rightarrow \times}$ . Indeed, as for the  $\lambda$ -calculus with products, the values are exactly the terms of form  $\lambda x.t$  or  $\langle t, u \rangle$ . However, in the case of constructor rewriting (Def. 2.1.2), we would like to build values from constructors. This is particularly useful with inductive types [BJO02]. According to Def. 4.1.4, we have to make them observable. To this end, we introduce appropriate *destructors*  $d \in \mathcal{D}$  in elimination contexts. To each  $c \in \mathcal{C}$  of type  $(\vec{T}, B)$  with  $|\vec{T}| > 0$  and each  $i \in \{1, \dots, |\vec{T}|\}$ , we associate a new unary *destructor symbol*  $d_{c,i} \in \mathcal{D}$  defined by the rewrite rule

$$d_{c,i}(c(x_1, \dots, x_n)) \mapsto_{\mathcal{D}} x_i .$$

Let  $\mathcal{U}$  be a new nullary symbol. For the elimination of a nullary constructor  $c$ , we use a new unary destructor  $d_c \in \mathcal{D}$  defined by the rewrite rule

$$d_c(c) \mapsto_{\mathcal{D}} \mathcal{U} .$$

**Lemma 4.1.5** Given a rewrite system  $\mathcal{R}$  with constructors in  $\mathcal{C}$ , let  $\mathcal{E}_{\Rightarrow \times \mathcal{C}}$  be the set of terms defined by the grammar

$$E[\ ] \in \mathcal{E}_{\Rightarrow \times \mathcal{C}} ::= [\ ] \mid E[\ ]t \mid \pi_1 E[\ ] \mid \pi_2 E[\ ] \mid d(E[\ ]) ,$$

where  $d \in \mathcal{D}$ . Then,  $\mathcal{E}_{\Rightarrow \times \mathcal{C}}$  is a set of evaluation contexts for  $\rightarrow_{\beta\pi\mathcal{R}\mathcal{D}}$ . The values for  $\rightarrow_{\beta\pi\mathcal{R}\mathcal{D}}$  in  $\mathcal{E}_{\Rightarrow \times \mathcal{C}}$  are exactly the terms of the form

$$\lambda x.t \quad \text{or} \quad \langle t_1, t_2 \rangle \quad \text{or} \quad c(t_1, \dots, t_n) \quad \text{where} \quad c \in \mathcal{C} .$$

**Example 4.1.6** Consider the system presented at Ex. 2.1.4. Its values for  $\rightarrow_{\beta\pi\mathcal{R}\mathcal{D}}$  in  $\mathcal{E}_{\Rightarrow \times \mathcal{C}}$  are the terms of the form

$$\lambda x.t \quad \text{or} \quad \langle t_1, t_2 \rangle \quad \text{or} \quad S(t) \quad \text{or} \quad 0 .$$

Indeed, we have

$$(\lambda x.t)u \rightarrow_{\beta} t[u/x] \quad \pi_i \langle t_1, t_2 \rangle \rightarrow_{\pi} t_i \quad d_{S,1}(S(t)) \rightarrow_{\mathcal{D}} t \quad \text{and} \quad d_0(0) \rightarrow_{\mathcal{D}} \mathcal{U} .$$

Note that values are preserved by reduction in all our examples.



**Some consequences of non-interaction.** In this paragraph, we explore some basic consequences of non-interaction on the strong normalizability of neutral terms in evaluation contexts.

First, the non-interaction of neutral terms with evaluation contexts is sufficient, together with the assumption that evaluation contexts are stable by reduction, to ensure the strong normalization of a neutral term  $t$  plugged in a strongly normalizing elimination context  $E[\ ]$ , as soon as  $E[u]$  is strongly normalizing for all *one-step* reduct  $u$  of  $t$ . This property is crucial for reducibility candidates.

**Lemma 4.1.7** *Let  $\rightarrow_R$  be a rewrite relation,  $E[\ ] \in \mathcal{E} \cap \mathcal{SN}_R$  be an evaluation context for  $\rightarrow_R$ , and let  $t \in \mathcal{N}_{RE}$  be a neutral term. Then,*

$$(\forall u. \ t \rightarrow_R u \implies E[u] \in \mathcal{SN}_R) \implies E[t] \in \mathcal{SN}_R.$$

PROOF. We have to show that  $v \in \mathcal{SN}_R$  for all  $v$  such that  $E[t] \rightarrow_R v$ . We reason by induction on  $E[\ ] \in \mathcal{SN}_R$ . Let  $v$  such that  $E[t] \rightarrow_R v$ . Since  $t$  is neutral, we have  $v = E'[t']$  with  $(E[\ ], t) \rightarrow_R (E'[\ ], t')$ , and there are two cases.

**Case of  $E[\ ] \rightarrow_R E'[\ ]$ .** We have  $E'[\ ] \in \mathcal{E}$  since  $\mathcal{E}$  is stable by reduction and  $E'[\ ] \in \mathcal{SN}_R$  since  $E[\ ] \in \mathcal{SN}_R$ . For all  $u \in (t)_R$ , since  $E[u] \rightarrow_R E'[u]$  and  $E[u] \in \mathcal{SN}_R$ , we have  $E'[u] \in \mathcal{SN}_R$ . Hence, we can apply the induction hypothesis on  $E'[\ ]$  and we conclude that  $E'[t] \in \mathcal{SN}_R$ .

**Case of  $t \rightarrow_R t'$ .** In this case, we have  $E[t'] \in \mathcal{SN}_R$  by assumption.  $\square$

Note that the induction principle of the above proof uses the hypothesis that evaluation contexts are stable by reduction.

We now come back to the assumption that evaluation contexts do not bind variables. Recall that this follows from the definition  $E[t] =_{\text{def}} (E[\ ])[t/[ \ ]]$ , which uses the capture-avoiding substitution. To allow variable capture in evaluation contexts, it is therefore sufficient to let  $E[t]$  be the textual replacement of  $[ \ ]$  by  $t$  in  $E[\ ]$ . As shown by the following example, borrowed from [vRS95], this may cause subtle interactions between contexts and some open terms that would usually else be neutral.

**Example 4.1.8 (Binding Evaluation Contexts – [vRS95])** Recall from Ex. 3.5.1 that  $\delta \notin \mathcal{SN}_\beta$  where  $\delta = \lambda x.x x$ , and consider the following set of evaluation contexts for  $\rightarrow_\beta$ :

$$\mathcal{E}_{\text{bind}} =_{\text{def}} \mathcal{E}_{\Rightarrow} \cup \{E[(\lambda y.[ \ ]) \delta] \mid E[\ ] \in \mathcal{E}_{\Rightarrow}\}$$

Given  $E[\ ] \in \mathcal{E}_{\text{bind}}$ , let  $E[t]$  be the textual replacement of  $[ \ ]$  by  $t$  in  $E[\ ]$ .

Let

$$C[\ ] =_{\text{def}} (\lambda y.[ \ ]) \delta \quad \text{and} \quad t =_{\text{def}} (\lambda x.z)(y y)$$

As expected, the term  $t$  is neutral in  $\mathcal{E}_{\Rightarrow}$ , but it is not in  $\mathcal{E}_{\text{bind}}$  because

$$C[t] = (\lambda y.(\lambda x.z)(y y)) \delta \rightarrow_\beta (\lambda x.z)(\delta \delta)$$

while  $(\lambda x.z)(\delta \delta)$  is not of the form  $C'[t']$  with  $C'[\ ] \in \mathcal{E}_{\text{bind}}$  and  $(C[\ ], t) \rightarrow_\beta (C'[\ ], t')$ .

This interaction between  $t$  and  $C[\ ]$  is critical since the property of Lem. 4.1.7 would fail if  $t$  were neutral in  $\mathcal{E}_{\text{bind}}$ . Indeed, we have  $(t)_\beta = \{z\}$ , and since  $C[z] = (\lambda y.z)\delta \in \mathcal{SN}_\beta$ , we get

$$\forall v. \ t \rightarrow_\beta v \implies C[v] \in \mathcal{SN}_\beta$$

However,

$$C[t] = (\lambda y.(\lambda x.z)(y y)) \delta \rightarrow_\beta (\lambda x.z)(\delta \delta) \notin \mathcal{SN}_\beta$$

Note that if we have defined  $C[u]$  as  $C[\ ] [u/\ ]$ , using the capture-avoiding substitution instead of the textual replacement, then  $t$  would have been neutral in  $\mathcal{E}_{bind}$ , but this would not have break Lem 4.1.7 since  $C[t]$  would have been the term  $(\lambda y'.(\lambda x.z) (y y)) \delta$  with  $y' \neq y$ .

Now, recall the property  $\mathcal{SN}_\beta \in \mathcal{SAT}_\beta$  (resp.  $\mathcal{SN}_{\beta\pi} \in \mathcal{SAT}_{\beta\pi}$ ) proved in Lem. 3.2.9 (resp. Lem. 3.3.2). This property holds because, for the  $\lambda$ -calculus with product, we can define a strong-normalization preserving notion of weak-head expansion. Namely, we have shown that given  $E[\ ] \in \mathcal{E}_{\Rightarrow \times}$ , we have

$$\begin{aligned} (t_2 \in \mathcal{SN}_{\beta\pi} \quad \wedge \quad E[t_1[t_2/x]] \in \mathcal{SN}_{\beta\pi}) &\implies E[(\lambda x.t_1) t_2] \in \mathcal{SN}_{\beta\pi}, \\ (t_{3-i} \in \mathcal{SN}_{\beta\pi} \quad \wedge \quad E[t_i] \in \mathcal{SN}_{\beta\pi}) &\implies E[\pi_i \langle t_1, t_2 \rangle] \in \mathcal{SN}_{\beta\pi}. \end{aligned}$$

The non-interaction of neutral terms with evaluation contexts is sufficient to show that the two properties above still hold when  $E[\ ] \in \mathcal{E}$  is any evaluation context such that  $(\lambda x.t_1)t_2$  and  $\pi_i \langle t_1, t_2 \rangle$  are neutral for  $\rightarrow_{\beta\pi}$  in  $\mathcal{E}$ . It is interesting to show this in a slightly more general framework.

**Lemma 4.1.9** *Let  $\rightarrow_R$  be a rewrite relation on  $\Lambda(\Sigma)$ .*

- (i) *Let  $\mathcal{E}$  be a set of elimination contexts for  $\rightarrow_{\beta R}$  such that  $(\lambda x.t)u \in \mathcal{N}_{\beta R \mathcal{E}}$  for all  $t, u \in \Lambda(\Sigma)$ . For all  $E[\ ] \in \mathcal{E}$ , if  $E[t[u/x]] \in \mathcal{SN}_{\beta R}$  and  $u \in \mathcal{SN}_{\beta R}$  then  $E[(\lambda x.t)u] \in \mathcal{SN}_{\beta R}$ .*
- (ii) *Let  $\mathcal{E}$  be a set of elimination contexts for  $\rightarrow_{\pi R}$  such that  $\pi_i \langle t_1, t_2 \rangle \in \mathcal{N}_{\pi R \mathcal{E}}$  for all  $t_1, t_2 \in \Lambda(\Sigma)$  and all  $i \in \{1, 2\}$ . For all  $E[\ ] \in \mathcal{E}$ , if  $E[t_i] \in \mathcal{SN}_{\pi R}$  and  $t_{3-i} \in \mathcal{SN}_{\pi R}$  then  $E[\pi_i \langle t_1, t_2 \rangle] \in \mathcal{SN}_{\pi R}$ .*

PROOF. We only show (i) because (ii) is similar and simpler.

- (i) Assume that  $E[t[u/x]], u \in \mathcal{SN}_{\beta R}$ . We must show that  $E[(\lambda x.t)u] \in \mathcal{SN}_{\beta R}$ , hence that for all  $v$ , if  $E[(\lambda x.t)u] \rightarrow_{\beta R} v$  then  $v \in \mathcal{SN}_{\beta R}$ .

We reason as in Lem. 3.2.9, using the non-interaction of  $(\lambda x.t)u$  with  $E[\ ]$ , which follows from the assumption that  $(\lambda x.t)u$  is neutral in  $\mathcal{E}$  for  $\rightarrow_{\beta R}$ , and weak standardization (Lem 3.2.8).

We have  $E[\ ], t \in \mathcal{SN}_{\beta R}$  since  $E[t[u/x]] \in \mathcal{SN}_{\beta R}$  and since  $E[\ ]$  contains at least one occurrence of  $[ \ ]$ . We reason by induction on tuples  $(E[\ ], t, u)$  ordered by the product extension of  $\rightarrow_{\beta R}$ .

Let  $v$  such that  $E[(\lambda x.t)u] \rightarrow_{\beta R} v$ . By non-interaction, there are two cases.

- $v = E'[(\lambda x.t)u]$  with  $E[\ ] \rightarrow_{\beta R} E'[\ ]$ . In this case, we conclude by induction hypothesis. Note that  $E'[t[u/x]] \in \mathcal{SN}_{\beta R}$  since  $E[\ ] \rightarrow_{\beta R} E'[\ ]$ .
- $v = E[s]$  with  $(\lambda x.t)u \rightarrow_{\beta R} s$ . By Lem 3.2.8, there are two subcases.
  - $s = (\lambda x.t')u'$  with  $(t, u) \rightarrow_{\beta R} (t', u')$ . In this case, we conclude by induction hypothesis. Note that  $E[t'[u'/x]] \in \mathcal{SN}_{\beta R}$  since  $(t, u) \rightarrow_{\beta R} (t', u')$ .
  - $s = t[u/x]$ . Since  $E[t[u/x]] \in \mathcal{SN}_{\beta R}$  by assumption. □

## 4.2 Reducibility Candidates

The key property of reducibility candidates w.r.t. neutral terms is the *neutral term property* (22): if  $C$  is a reducibility candidate and  $t$  is a neutral term, then

$$(\forall v. \quad t \rightarrow_R v \implies v \in C) \implies t \in C.$$

In particular, every reducibility candidate contains all neutral terms in normal form.

**Elimination contexts.** In order to get a reducibility family in the sense of Def. 2.2.7, we need that all reducibility candidate contains the variables. In the case of usual reducibility candidates, this follows from property (24) which ensures that variables are neutral terms. In order to generalize this notion, we assume that evaluation contexts do not interact with variables. This amounts to assuming that all variables are neutral terms. Since variables are moreover assumed to be in normal form (by Def. 2.1.1 of rewrite relations), it follows that they belong to every reducibility candidate.

Evaluation contexts must satisfy another interesting property, which is used to ensure that the function space  $\_ \Rightarrow \_$  and the product space  $\_ \times \_$  preserve reducibility candidates (this will actually be shown in Prop. 4.4.5). Let us consider the case of products. As seen above,  $\mathcal{E}_\times$  is a set of evaluation contexts for  $\rightarrow_\pi$ . Moreover, the neutral term property writes

$$(\forall v. \ t \rightarrow_\pi v \implies v \in C) \implies t \in C. \quad (25)$$

Suppose that we want to show that if both  $A_1$  and  $A_2$  are reducibility candidates, then  $A_1 \times A_2$  is also a reducibility candidate. In particular, we have to prove that  $A_1 \times A_2$  satisfies property (25), as soon as  $A_1$  and  $A_2$  also do. So let  $t$  be a neutral term for  $\rightarrow_\pi$  in  $\mathcal{E}_\pi$  such that  $(t)_\pi \in A_1 \times A_2$ . By definition of  $A_1 \times A_2$ , this means that  $\pi_i v \in A_i$  for all  $v \in (t)_\pi$  and all  $i \in \{1, 2\}$ . Since  $\pi_i [ ]$  is an evaluation context, and since  $t$  is neutral, we know that

$$(\pi_i t)_\pi = \{ \pi_i v \mid v \in (t)_\pi \},$$

and it follows that  $(\pi_i t)_\pi \subseteq A_i$ . Now, how can we conclude ? By using that the term  $\pi_i t$  is also neutral. Hence, by applying property (25) to  $A_i$ , since  $(\pi_i t)_\pi \subseteq A_i$ , we have  $\pi_i t \in A_i$ . By definition of  $A_1 \times A_2$ , we deduce that  $t \in A_1 \times A_2$ .

The key-point of the above reasoning was to use an instance of the following property:

$$(t \in \mathcal{N}_\pi \ \wedge \ E[ ] \in \mathcal{E}_\pi) \implies E[t] \in \mathcal{N}_\pi.$$

Therefore, we are interested in evaluation contexts such that

$$(t \in \mathcal{N}_{\mathcal{R}\mathcal{E}} \ \wedge \ E[ ] \in \mathcal{E}) \implies E[t] \in \mathcal{N}_{\mathcal{R}\mathcal{E}}.$$

In words, neutral terms have to be closed by composition with evaluation contexts. This leads to the notion of *elimination contexts*.

**Definition 4.2.1 (Elimination Contexts)** Let  $\mathcal{E}$  be a set of evaluation contexts for  $\rightarrow_{\mathcal{R}}$ . Then  $\mathcal{E}$  is a set of elimination contexts for  $\rightarrow_{\mathcal{R}}$  if

- (i) all variables are neutral:  $\mathcal{X} \subseteq \mathcal{N}_{\mathcal{R}\mathcal{E}}$ ,
- (ii) if  $t \in \mathcal{N}_{\mathcal{R}\mathcal{E}}$  and  $E[ ] \in \mathcal{E}$  then  $E[t] \in \mathcal{N}_{\mathcal{R}\mathcal{E}}$ .

All the evaluation contexts seen above are elimination contexts:

- $\mathcal{E}_{\Rightarrow}$  is a set of elimination contexts for  $\rightarrow_\beta$ ,
- $\mathcal{E}_\times$  is a set of elimination contexts for  $\rightarrow_\pi$ ,
- $\mathcal{E}_{\Rightarrow \times}$  is a set of elimination contexts for  $\rightarrow_{\beta\pi}$  and for  $\rightarrow_{\beta\pi\mathcal{R}}$ , where  $\mathcal{R}$  is a rewrite system,
- $\mathcal{E}_{\Rightarrow \times \mathcal{C}}$  is a set of elimination contexts for  $\rightarrow_{\beta\pi\mathcal{R}\mathcal{D}}$ , where  $\mathcal{R}$  is a rewrite system with constructors in  $\mathcal{C}$ .

**Reducibility candidates.** We now define reducibility candidates in the usual way [GLT89, Gal89]: our generalization regards neutral terms and their definition using elimination contexts. Assume that  $\mathcal{E}$  is a set of elimination contexts for  $\rightarrow_R$ .

**Definition 4.2.2 (Reducibility Candidates)** *The set  $\mathcal{CR}_{\mathcal{R}\mathcal{E}}$  of reducibility candidates for  $\rightarrow_R$  in  $\mathcal{E}$  is the set of all  $C \subseteq \mathcal{SN}_R$  such that*

(CR0) *if  $t \in C$  and  $t \rightarrow_R u$  then  $u \in C$ ,*

(CR1) *if  $t \in \mathcal{N}_{\mathcal{R}\mathcal{E}}$  and  $\forall u. t \rightarrow_R u \implies u \in C$  then  $t \in C$ .*

The clause (CR1) is the neutral term property.

**Example 4.2.3** *For the pure  $\lambda$ -calculus and the  $\lambda$ -calculus with products, if we take as elimination contexts respectively  $\mathcal{E}_{\Rightarrow}$  and  $\mathcal{E}_{\Rightarrow \times}$ , then our definition correspond to the usual one (see for instance [GLT89]). Moreover,*

- *we let  $\mathcal{CR}_{\beta}$  be the set of reducibility candidates for  $\rightarrow_{\beta}$  in  $\mathcal{E}_{\Rightarrow}$ ,*
- *we let  $\mathcal{CR}_{\beta\pi}$  be the set of reducibility candidates for  $\rightarrow_{\beta\pi}$  in  $\mathcal{E}_{\Rightarrow \times}$ ,*
- *given a rewrite system  $\mathcal{R}$ , we let  $\mathcal{CR}_{\beta\pi\mathcal{R}}$  (resp.  $\mathcal{CR}_{\beta\mathcal{R}}$ ) be the set of reducibility candidates for  $\rightarrow_{\beta\pi\mathcal{R}}$  in  $\mathcal{E}_{\Rightarrow \times}$  (resp. for  $\rightarrow_{\beta\mathcal{R}}$  in  $\mathcal{E}_{\Rightarrow}$ ); if moreover  $\mathcal{C}$  is a set of constructors for  $\mathcal{R}$ , then we let  $\mathcal{CR}_{\beta\pi\mathcal{R}\mathcal{C}}$  (resp.  $\mathcal{CR}_{\beta\mathcal{R}\mathcal{C}}$ ) be the set of reducibility candidates for  $\rightarrow_{\beta\pi\mathcal{R}\mathcal{D}}$  in  $\mathcal{E}_{\Rightarrow \times \mathcal{C}}$  (resp. for  $\rightarrow_{\beta\mathcal{R}\mathcal{D}}$  in  $\mathcal{E}_{\Rightarrow \mathcal{C}}$ ).*

As we have done with saturated sets in Sect. 3, in order to show that  $\mathcal{CR}_{\mathcal{R}\mathcal{E}}$  is not empty, we show that  $\mathcal{SN}_R \in \mathcal{CR}_{\mathcal{R}\mathcal{E}}$ . Since it is clear that  $\mathcal{SN}_R$  is stable by reduction, it remains to show that  $\mathcal{SN}_R$  satisfies the neutral term property. But this is trivial, since  $\mathcal{SN}_R$  is the smallest set such that  $(t)_R \subseteq \mathcal{SN}_R$  implies  $t \in \mathcal{SN}_R$ . We thus have

**Lemma 4.2.4**  $\mathcal{SN}_R \in \mathcal{CR}_{\mathcal{R}\mathcal{E}}$ .

As said at the beginning of this section, Girard's reducibility candidates rely on the fact that since neutral terms do not interact with elimination contexts, their properties w.r.t. reducibility can be shown by induction on  $\mathcal{SN}_R$ . This is property (23), shown in Lem. 4.2.5 below, which extends Lem. 4.1.7 to all reducibility candidates when elimination contexts are used in place of evaluation contexts.

**Lemma 4.2.5** *Let  $t \in \mathcal{N}_{\mathcal{R}\mathcal{E}}$  and  $E[\ ] \in \mathcal{E} \cap \mathcal{SN}_R$ . Then for all  $C \in \mathcal{CR}_{\mathcal{R}\mathcal{E}}$ ,*

$$(\forall u. t \rightarrow_R u \implies E[u] \in C) \implies E[t] \in C.$$

PROOF. First, since  $t \in \mathcal{N}_{\mathcal{R}\mathcal{E}}$  and  $E[\ ] \in \mathcal{E}$ , we have  $E[t] \in \mathcal{N}_{\mathcal{R}\mathcal{E}}$  by Def. 4.2.1.(ii). Hence we only have to show that  $(E[t])_R \subseteq C$ .

We reason by induction on  $E[\ ] \in \mathcal{SN}_R$ . Let  $v$  such that  $E[t] \rightarrow_R v$ . Since  $t$  is neutral, we have  $v = E'[t']$  with  $(E[\ ], t) \rightarrow_R (E'[\ ], t')$ , and there are two cases.

**Case of  $E[\ ] \rightarrow_R E'[\ ]$ .** We have  $E'[\ ] \in \mathcal{E}$  by Def. 4.1.1. and  $E'[\ ] \in \mathcal{SN}_R$  since  $E[\ ] \in \mathcal{SN}_R$ . For all  $u \in (t)_R$ , since  $E[u] \rightarrow_R E'[u]$  and  $E[u] \in C$ , we have  $E'[u] \in C$  by (CR0). Hence we can apply the induction hypothesis on  $E'[\ ]$  and conclude that  $E'[t] \in C$ .

**Case of  $t \rightarrow_R t'$ .** In this case we have  $E[t'] \in C$  by assumption. □

In particular, given a rewrite system  $\mathcal{R}$  with constructors in  $\mathcal{C}$ , reducibility candidates in  $\mathcal{CR}_{\beta\pi\mathcal{R}\mathcal{C}}$  satisfy property (21) for non-constructor symbols  $f \in \Sigma \setminus \mathcal{C}$ . Moreover, reducibility candidates  $C \in \mathcal{CR}_{\beta\pi}$  satisfy clauses (SAT0), (SAT2 $_{\beta}$ ) and (SAT2 $_{\pi i}$ ). This property is already known for the pure  $\lambda$ -calculus [Gal89] and for the  $\lambda$ -calculus with products [Luo90]. Our formulation of reducibility candidates allows to formulate it in a more general and modular way. Thanks to the neutral term property, the proof is actually the same as for Lem. 4.1.9.

**Lemma 4.2.6** *Let  $\rightarrow_{\mathcal{R}}$  be a rewrite relation on  $\Lambda(\Sigma)$ .*

(SAT1) *Let  $\mathcal{E}$  be a set of elimination contexts for  $\rightarrow_{\mathcal{R}}$ .*

*For all  $C \in \mathcal{CR}_{\mathcal{R}\mathcal{E}}$ , if  $E[\ ] \in \mathcal{E} \cap \mathcal{SN}_{\mathcal{R}}$  and  $x \in \mathcal{X}$  then  $E[x] \in C$ .*

(SAT2 $_{\beta}$ ) *Let  $\mathcal{E}$  be a set of elimination contexts for  $\rightarrow_{\beta\mathcal{R}}$  such that  $(\lambda x.t)u \in \mathcal{N}_{\beta\mathcal{R}\mathcal{E}}$  for all  $t, u \in \Lambda(\Sigma)$ .*

*For all  $C \in \mathcal{CR}_{\beta\mathcal{R}\mathcal{E}}$  and all  $E[\ ] \in \mathcal{E}$ , if  $E[t[u/x]] \in C$  and  $u \in \mathcal{SN}_{\beta\mathcal{R}}$  then  $E[(\lambda x.t)u] \in C$ .*

(SAT2 $_{\pi i}$ ) *Let  $\mathcal{E}$  be a set of elimination contexts for  $\rightarrow_{\pi\mathcal{R}}$  such that  $\pi_i(t_1, t_2) \in \mathcal{N}_{\pi\mathcal{R}\mathcal{E}}$  for all  $t_1, t_2 \in \Lambda(\Sigma)$  and all  $i \in \{1, 2\}$ .*

*For all  $C \in \mathcal{CR}_{\pi\mathcal{R}\mathcal{E}}$  and all  $E[\ ] \in \mathcal{E}$ , if  $E[t_i] \in C$  and  $t_{3-i} \in \mathcal{SN}_{\pi\mathcal{R}}$  then  $E[\pi_i(t_1, t_2)] \in C$ .*

PROOF.

(SAT1). By definition, the terms of the form  $E[x]$  with  $E[\ ] \in \mathcal{E}$  satisfy

$$\forall v. \ E[x] \rightarrow_{\mathcal{R}} v \implies (v = E'[x] \text{ with } E'[\ ] \in \mathcal{E} \text{ and } E[\ ] \rightarrow_{\mathcal{R}} E'[\ ]).$$

There are thus neutral terms whose reducts are all neutrals. We conclude by induction on  $E[\ ] \in \mathcal{SN}_{\mathcal{R}}$ .

(SAT2 $_{\beta}$ ) and (SAT2 $_{\pi i}$ ). We only show the case of (SAT2 $_{\beta}$ ) because (SAT2 $_{\pi i}$ ) is similar and simpler.

We have to show that if  $E[t[u/x]] \in C$  with  $u \in \mathcal{SN}_{\beta\mathcal{R}}$  then  $E[(\lambda x.t)u] \in C$ . Since  $E[(\lambda x.t)u] \in \mathcal{N}_{\beta\mathcal{R}\mathcal{E}}$ , by (CR1) it is sufficient to show that  $E[(\lambda x.t)u] \rightarrow_{\beta\mathcal{R}} v$  implies  $v \in C$ .

We can therefore reason as in Lem. 4.1.9, using the non-interaction of  $(\lambda x.t)u$  with  $E[\ ]$ , which follows from the assumption that  $(\lambda x.t)u$  is neutral in  $\mathcal{E}$  for  $\rightarrow_{\beta\mathcal{R}}$ , and weak standardization (Lem. 3.2.8).

We have  $E[\ ], t \in \mathcal{SN}_{\beta\mathcal{R}}$  since  $E[t[u/x]] \in \mathcal{SN}_{\beta\mathcal{R}}$  and since  $E[\ ]$  contains at least one occurrence of  $[\ ]$ . We reason by induction on tuples  $(E[\ ], t, u)$  ordered by the product extension of  $\rightarrow_{\beta\mathcal{R}}$ .

Let  $v$  such that  $E[(\lambda x.t)u] \rightarrow_{\beta\mathcal{R}} v$ . By non-interaction, there are two cases.

- $v = E'[(\lambda x.t)u]$  with  $E[\ ] \rightarrow_{\beta\mathcal{R}} E'[\ ]$ . In this case, we conclude by induction hypothesis. Note that  $E'[t[u/x]] \in C$  thanks to (CR0).
- $v = E[s]$  with  $(\lambda x.t)u \rightarrow_{\beta\mathcal{R}} s$ . By Lem 3.2.8, there are two subcases.
  - $s = (\lambda x.t')u'$  with  $(t, u) \rightarrow_{\beta\mathcal{R}} (t', u')$ . In this case, we conclude by induction hypothesis. Note that  $E[t'[u'/x]] \in C$  thanks to (CR0).
  - $s = t[u/x]$ . Since  $E[t[u/x]] \in \mathcal{SN}_{\beta\mathcal{R}}$  by assumption. □

### 4.3 The Closure Operator of Reducibility Candidates

Because  $\rightarrow_R$  needs not to be finitely branching, the closure operator of reducibility candidates is not always definable by induction on  $\mathbb{N}$ . This is in contrast with  $\mathcal{SAT}(\_)$ . Recall that we only consider countable signatures. It follows that  $\Lambda(\Sigma)$  is a countable set, hence that the closure operator of  $\mathcal{CR}$  can be defined by induction on countable ordinals. Therefore, we consider a well-ordered set  $(\mathfrak{D}, \leq)$  satisfying the axioms of the countable ordinals, as presented for instance in [Gal91].

Let  $\rightarrow_R$  be a rewrite relation on  $\Lambda(\Sigma)$  and  $\mathcal{E}$  be a set of elimination contexts for  $\rightarrow_R$ .

#### Definition 4.3.1

— We define the function  $\mathcal{CR} : \mathcal{P}(\mathcal{SN}_R) \mapsto \mathcal{P}(\mathcal{SN}_R)$  as follows:

$$\mathcal{CR}(X) \quad =_{def} \quad X \cup \{t \in \mathcal{N}_{R\mathcal{E}} \mid \forall u. \quad t \rightarrow_R u \implies u \in X\}.$$

— We define the functions  $\mathcal{CR}_a : \mathcal{P}(\mathcal{SN}_R) \mapsto \mathcal{P}(\mathcal{SN}_R)$  by induction on  $a \in \mathfrak{D}$  as follows:

$$\begin{aligned} \mathcal{CR}_0(X) &=_{def} (X)_R^* \\ \mathcal{CR}_{a+1}(X) &=_{def} \mathcal{CR}(\mathcal{CR}_a(X)) \\ \mathcal{CR}_\lambda(X) &=_{def} \bigcup_{a < \lambda} \mathcal{CR}_a(X) \quad \text{if } \lambda \text{ is a limit ordinal.} \end{aligned}$$

It is clear that  $\mathcal{CR}$  is a monotone function on  $\mathcal{P}(\mathcal{SN}_R)$ . It follows that given  $X \in \mathcal{P}(\mathcal{SN}_R)$ ,  $\mathcal{CR}$  is monotone on the complete lattice  $\{(Y)_R^* \mid X \subseteq Y \subseteq \mathcal{SN}_R\}$ , on which it therefore has a least fixpoint. We now check that this fixpoint is  $\mathcal{CR}_{p_X}(X)$  for some countable ordinal  $p_X \in \mathfrak{D}$ .

**Proposition 4.3.2** *Given  $X \subseteq \mathcal{SN}_R$ ,*

- (i) *for all  $a, b \in \mathfrak{D}$ , if  $a \leq b$  then  $\mathcal{CR}_a(X) \subseteq \mathcal{CR}_b(X)$ ,*
- (ii) *there is  $p_X \in \mathfrak{D}$  such that  $\mathcal{CR}_{p_X}(X)$  is the least fixpoint of  $\mathcal{CR}$  in  $\{(Y)_R^* \mid X \subseteq Y \subseteq \mathcal{SN}_R\}$ ,*
- (iii) *we have  $\mathcal{CR}_{p_X}(X) = \mathcal{CR}_a(X)$  whenever  $p_X \leq a \in \mathfrak{D}$ .*

PROOF.

- (i) By induction on  $b \in \mathfrak{D}$ .
- (ii) Assume that the property does not hold. Then, by (i) we have  $\mathcal{CR}_a(X) \subsetneq \mathcal{CR}_b(X)$  for all  $a, b \in \mathfrak{D}$  such that  $a < b$ . Since  $\mathfrak{D}$  is not countable, we would have an uncountable set  $\{t_b \mid b \in \mathfrak{D}\} \subseteq \Lambda(\Sigma)$ . But this is not possible since  $\Lambda(\Sigma)$  is countable. It follows that the least fixpoint of  $\mathcal{CR}$  in  $\{(Y)_R^* \mid X \subseteq Y \subseteq \mathcal{SN}_R\}$  is  $\mathcal{CR}_{p_X}(X)$  for some  $p_X \in \mathfrak{D}$ .
- (iii) By induction on  $a$ .

□

An important point is that  $t \in \mathcal{CR}_{p_X}(X)$  if and only if  $t \in \mathcal{CR}_a(X)$  for some non-limit ordinal  $a < p_X$ . In other words, if  $t \in \mathcal{CR}_a(X)$  with  $a$  as small as possible, then  $a$  is either 0 or of the form  $b + 1$ . Indeed, if  $t \in \mathcal{CR}_\lambda(X)$  for a limit ordinal  $\lambda$ , then by definition of  $\mathcal{CR}_\lambda(X)$  there is some  $a < \lambda$  such that  $t \in \mathcal{CR}_a(X)$ .

**Definition 4.3.3** *We define the function  $\mathcal{CR} : \mathcal{P}(\mathcal{SN}_R) \mapsto \mathcal{P}(\mathcal{SN}_R)$  as  $\mathcal{CR}(X) =_{def} \mathcal{CR}_{p_X}(X)$ , where  $\mathcal{CR}_{p_X}(X)$  is the least fixpoint of  $\mathcal{CR}$  in  $\{(Y)_R^* \mid X \subseteq Y \subseteq \mathcal{SN}_R\}$ .*

We now check that  $\mathcal{CR}$  is indeed the closure operator which defines reducibility candidates.

**Lemma 4.3.4** *If  $X \subseteq \mathcal{SN}_R$  then  $\mathcal{CR}(X)$  is the least reducibility candidate containing  $X$ .*

PROOF. We begin by showing that  $\mathcal{CR}(X) \in \mathcal{CR}_{\mathcal{RE}}$  for all  $X \subseteq \mathcal{SN}_R$ . First, since  $\mathcal{CR}$  is a map on  $\mathcal{P}(\mathcal{SN}_R)$ , it is clear that  $\mathcal{CR}(X) \subseteq \mathcal{SN}_R$ . We now check the clauses  $(\mathcal{CR}0)$  and  $(\mathcal{CR}1)$ .

$(\mathcal{CR}0)$  Let  $t \rightarrow_R u$  and  $t \in \mathcal{CR}_a(X)$  with  $a$  as small as possible. If  $a = 0$ , then we have  $u \in (X)_R^* = \mathcal{CR}_0$ . Otherwise,  $a = b + 1$ ,  $t \in \mathcal{N}_{\mathcal{RE}}$  and  $u \in \mathcal{CR}_b(X)$  by definition.

$(\mathcal{CR}1)$  Let  $t \in \mathcal{N}_{\mathcal{RE}}$  with  $(t)_R \subseteq \mathcal{CR}(X)$ . There is  $a < p_X$  such that  $(t)_R \subseteq \mathcal{CR}_a(X)$ , hence  $t \in \mathcal{CR}_{a+1}(X)$ .

We now show by induction on  $a$  that  $\mathcal{CR}_a(X) \subseteq C$  for all  $C \in \mathcal{CR}_{\mathcal{RE}}$  containing  $X$ . By  $(\mathcal{CR}0)$ , we have  $\mathcal{CR}_0(X) = (X)_R^* \subseteq C$ . If  $t \in \mathcal{CR}_{a+1}(X) \setminus \mathcal{CR}_a(X)$  then  $t \in \mathcal{N}_{\mathcal{RE}}$  and by induction hypothesis  $(t)_R \subseteq \mathcal{CR}_a(X) \subseteq C$ , hence  $t \in C$  by  $(\mathcal{CR}1)$ . Moreover, if  $\lambda$  is a limit ordinal, then by induction hypothesis we have  $\mathcal{CR}_a(X) \subseteq C$  for all  $a < \lambda$ , hence  $\mathcal{CR}_\lambda(X) \subseteq C$ .  $\square$

It follows that  $\mathcal{CR}_{\mathcal{RE}} = \{\mathcal{CR}(X) \mid X \subseteq \mathcal{SN}_R\}$ . Let us now show that  $\mathcal{CR}(\cdot)$  is a closure operator.

**Lemma 4.3.5**  *$\mathcal{CR} : \mathcal{P}(\mathcal{SN}_R) \mapsto \mathcal{P}(\mathcal{SN}_R)$  is a closure operator.*

PROOF.

- Idempotency. By Lem. 4.3.4.
- Extensivity. Let  $X \in \mathcal{P}(\mathcal{SN}_R)$ . By induction on  $a \in \mathfrak{D}$ , we have  $X \subseteq \mathcal{CR}_a(X)$  for all  $a \in \mathfrak{D}$ , hence  $X \subseteq \mathcal{CR}(X) = \mathcal{CR}_{p_X}(X)$ .
- Monotony. Let  $X \subseteq Y \in \mathcal{P}(\mathcal{SN}_R)$ . By induction on  $a \in \mathfrak{D}$ , we have  $\mathcal{CR}_a(X) \subseteq \mathcal{CR}_a(Y)$  for all  $a \in \mathfrak{D}$ . It follows that  $\mathcal{CR}(X) = \mathcal{CR}_{p_X}(X) \subseteq \mathcal{CR}_{p_X}(Y)$ .  
It remains to show that  $\mathcal{CR}_{p_X}(Y) \subseteq \mathcal{CR}_{p_Y}(Y)$ . This follows from Prop. 4.3.2.(i) if  $p_X \subseteq p_Y$ , and otherwise we have  $\mathcal{CR}_{p_X}(Y) = \mathcal{CR}_{p_Y}(Y)$  by Prop. 4.3.2.(iii).  $\square$

Since  $\mathcal{CR}_{\mathcal{RE}}$  is defined by the closure operator  $\mathcal{CR} : \mathcal{P}(\mathcal{SN}_R) \mapsto \mathcal{P}(\mathcal{SN}_R)$ , it follows from Lem. 2.2.6 that it is a complete lattice whose maximal element is  $\mathcal{SN}_R$ , and whose g.l.b.'s are given by intersections.

## 4.4 Hereditary Neutral Terms

In Sect. 3.2, we have seen that to get a type interpretation with saturated sets for the pure  $\lambda$ -calculus, we must ensure that  $A \Rightarrow B \in \mathcal{SAT}_\beta$  whenever  $A, B \in \mathcal{SAT}_\beta$ . In particular, this implies that  $A \Rightarrow B \subseteq \mathcal{SN}_\beta$ , hence that  $A \neq \emptyset$ . Moreover, showing that  $A \neq \emptyset$  amounts to showing that  $\mathcal{X} \subseteq A$ . Therefore, we have to show that each reducibility candidate contains  $\mathcal{X}$ . To this end, it is sufficient to show that the least reducibility candidate contains  $\mathcal{X}$ .

In this section, we define and prove some properties of this set, also called the set of hereditary neutral terms, and denoted by  $\mathcal{HN}_{\mathcal{RE}}$ , where  $\rightarrow_R$  is a rewrite relation and  $\mathcal{E}$  is a set of elimination contexts for  $\rightarrow_R$ .

**Definition 4.4.1 (Hereditary Neutral Terms)** *We define  $\mathcal{HN}_{\mathcal{RE}}$ , the set of hereditary neutral terms for  $\rightarrow_R$  in  $\mathcal{E}$  as the least set such that*

$$\forall t \in \mathcal{N}_{\mathcal{RE}}. \quad (\forall u. t \rightarrow_R u \implies u \in \mathcal{HN}_{\mathcal{RE}}) \implies t \in \mathcal{HN}_{\mathcal{RE}}.$$



In words, a term is hereditary neutral if and only if it is a strongly normalizing neutral terms whose reducts are neutrals. The closure operator  $\mathcal{CR}(\_)$  is a convenient tool to show that  $\mathcal{HN}_{\mathcal{RE}}$  is the least reducibility candidate.

**Proposition 4.4.2**  *$\mathcal{HN}_{\mathcal{RE}}$  is the least element of  $\mathcal{CR}_{\mathcal{RE}}$ .*

PROOF. By Lem. 2.2.6, the least element of  $\mathcal{CR}_{\mathcal{RE}}$  is the closure of the least element  $\emptyset$  of  $\mathcal{P}(\mathcal{SN}_{\mathcal{R}})$ . But we have  $\mathcal{HN}_{\mathcal{RE}} = \mathcal{CR}(\emptyset)$  since  $\mathcal{CR}(\emptyset)$  is the least set such that

$$\mathcal{CR}(\emptyset) = \mathcal{CR}(\emptyset) \cup \{t \in \mathcal{N}_{\mathcal{RE}} \mid \forall u. \quad t \rightarrow_{\mathcal{R}} u \implies u \in \mathcal{CR}(\emptyset)\}.$$

□

According to the discussion at the beginning of Sect. 4.2, rewrite relations and eliminations contexts are defined in such a way that variables are neutral terms in normal form. We thus have  $\mathcal{X} \subseteq \mathcal{HN}_{\mathcal{RE}}$ , and we deduce that  $\mathcal{X} \subseteq \mathcal{C}$  for all candidate  $\mathcal{C}$ .

We let  $\mathcal{HN}_{\beta}$  (resp.  $\mathcal{HN}_{\beta\pi}$ ) be the set of hereditary neutral terms for  $\rightarrow_{\beta}$  in  $\mathcal{E}_{\Rightarrow}$  (resp. for  $\rightarrow_{\beta\pi}$  in  $\mathcal{E}_{\Rightarrow \times}$ ). For the pure  $\lambda$ -calculus, recall that neutral terms are exactly the terms of the form

$$E[x] \quad \text{or} \quad E[(\lambda x.t)u] \quad \text{with} \quad E[\_] \in \mathcal{E}_{\Rightarrow}.$$

Hence  $\beta$ -normal neutral terms are of the form  $E[x]$ , and hereditary neutral terms are exactly the strongly normalizing terms which reduces to a term of the form  $E[x]$  (recall that values are preserved by reduction). Moreover,  $\mathcal{HN}_{\beta}$  is the least element of  $\mathcal{SAT}_{\beta}$ .

**Proposition 4.4.3**  *$\mathcal{HN}_{\beta} \subseteq S$  for all  $S \in \mathcal{SAT}_{\beta}$ .*

PROOF. Let  $S \in \mathcal{SAT}_{\beta}$ . We have to show that  $t \in \mathcal{HN}_{\beta}$  implies  $t \in S$ . By Prop. 4.4.2 and Prop. (ii), for all  $t \in \mathcal{HN}_{\beta}$ , there is a least  $\alpha \in \mathcal{D}$  such that  $t \in \mathcal{CR}_{\alpha}(\emptyset)$ . By induction on  $\alpha \in \mathcal{D}$ , we show that if  $\alpha$  is the least ordinal such that  $t \in \mathcal{CR}_{\alpha}(\emptyset)$ , then  $t \in S$ . Note  $\alpha$  is either 0 or a successor ordinal  $\beta + 1$ , and that the first case is not possible since  $\mathcal{CR}_0(\emptyset) = \emptyset$ . So assume that  $t \in \mathcal{CR}_{\beta+1}(\emptyset)$ . Then  $t \in \mathcal{N}_{\beta}$  and  $(t)_{\mathcal{R}} \subseteq \mathcal{CR}_{\beta}(\emptyset) \subseteq \mathcal{SN}_{\beta}$ . Since  $t$  is neutral, there are two cases.

- If  $t = E[x]$ , then  $E[\_] \in \mathcal{SN}_{\beta}$  since  $(t)_{\beta} \subseteq \mathcal{SN}_{\beta}$ , hence  $t \in S$  by  $(\mathcal{SAT}1)$ .
- Otherwise,  $t = E[(\lambda x.t_1)t_2]$ . We have  $t_2, E[t_1[t_2/x]] \in \mathcal{SN}_{\beta}$  since  $(t)_{\beta} \subseteq \mathcal{SN}_{\beta}$  and  $E[t_1[t_2/x]] \in S$  since  $\mathcal{CR}_{\beta}(\emptyset) \subseteq S$  by induction hypothesis. It follows that  $t \in S$  by  $(\mathcal{SAT}2_{\beta})$ . □

An interesting property of hereditary neutral terms is that they are stable by composition with strongly normalizing elimination contexts. This is a consequence of the assumption that neutral terms are stable by composition with elimination contexts.

**Proposition 4.4.4** *If  $t \in \mathcal{HN}_{\mathcal{RE}}$  and  $E[\_] \in \mathcal{E} \cap \mathcal{SN}_{\mathcal{R}}$  then  $E[t] \in \mathcal{HN}_{\mathcal{RE}}$ .*

PROOF. We reason by induction on pairs  $(E[\_], t)$  ordered by the product extension of  $\rightarrow_{\mathcal{R}}$ .

So, let  $t \in \mathcal{HN}_{\mathcal{RE}}$  and  $E[\_] \in \mathcal{E} \cap \mathcal{SN}_{\mathcal{R}}$ . Since  $E[t] \in \mathcal{N}_{\mathcal{RE}}$ , we have  $E[t] \in \mathcal{HN}_{\mathcal{RE}}$  whenever  $(E[t])_{\mathcal{R}} \subseteq \mathcal{HN}_{\mathcal{RE}}$ . Now, let  $v$  such that  $E[t] \rightarrow_{\mathcal{R}} v$ . Since  $t$  is neutral, it follows that  $v = E'[t']$  with  $(E[\_], t) \rightarrow_{\mathcal{R}} (E'[\_], t')$ . Moreover, we have  $t' \in \mathcal{HN}_{\mathcal{RE}}$  since  $t \in \mathcal{HN}_{\mathcal{RE}}$ . Thus  $v \in \mathcal{HN}_{\mathcal{RE}}$  by induction hypothesis. □

Since variables are hereditary neutral, it follows from Prop. 4.4.4 that  $E[x] \in \mathcal{HN}_{\mathcal{RE}}$  for all  $E[\_] \in \mathcal{E} \cap \mathcal{SN}_{\mathcal{R}}$  and all  $x \in \mathcal{X}$ . We can now conclude the discussion at the beginning of this section, and show that for pure  $\lambda$ -calculus we have  $A \Rightarrow B \in \mathcal{CR}_{\beta}$  for all  $A, B \in \mathcal{CR}_{\beta}$ . As with Lem. 4.1.9 and Lem. 4.2.6, it is interesting to show this in a slightly more general framework.



**Proposition 4.4.5**

(i) If  $[ ]t \in \mathcal{E}$  for all  $t \in \Lambda(\Sigma)$ , then

$$C_1, C_2 \in \mathcal{CR}_{\mathcal{RE}} \implies C_2 \Rightarrow C_1 \in \mathcal{CR}_{\mathcal{RE}} .$$

(ii) If  $\pi_i [ ] \in \mathcal{E}$  for all  $i \in \{1, 2\}$ , then

$$C_1, C_2 \in \mathcal{CR}_{\mathcal{RE}} \implies C_1 \times C_2 \in \mathcal{CR}_{\mathcal{RE}} .$$

PROOF. First, by using that  $\mathcal{X} \subseteq \mathcal{HN}_{\mathcal{RE}} \subseteq \mathcal{C}$  for all  $C \in \mathcal{CR}_{\mathcal{RE}}$ , we get  $C_2 \Rightarrow C_1, C_1 \times C_2 \subseteq \mathcal{SN}_{\mathcal{R}}$ , exactly as with saturated sets in Prop. 3.2.10 and Prop. 3.3.3.

We now check that  $C_2 \Rightarrow C_1$  and  $C_1 \times C_2$  satisfy the clauses (CR0) and (CR1). We only detail the case of  $- \Rightarrow -$ , because that of  $- \times -$  is similar and simpler.

(CR0) As in Prop. 3.5.2.

(CR1) Let  $t \in \mathcal{N}_{\mathcal{RE}}$  such that  $(t)_{\mathcal{R}} \subseteq C_2 \Rightarrow C_1$  and let  $a \in C_2$ . For all  $u \in (t)_{\mathcal{R}}$ , we have  $ua \in C_1$ . Since  $[ ]a \in \mathcal{SN}_{\mathcal{R}}$ , it follows from Lem. 4.2.5 that  $ta \in C_1$ . We conclude that  $t \in C_2 \Rightarrow C_1$ .  $\square$

We have now sufficient material to show that in the case of the  $\lambda$ -calculus with products,  $\mathcal{CR}_{\beta\pi}$  is a reducibility family which lead to an adequate type interpretation. Adequacy is obtained as in Lem. 3.3.4, by combining Lem. 4.2.6 with Prop. 4.4.5.

## 4.5 The Structure of Reducibility Candidates

In this section, we explore further the structure of reducibility candidates.

We begin by some general properties of closure operators on powersets. In particular, a closure operator  $\overline{(-)} : \mathcal{P}(D) \mapsto \mathcal{P}(D)$  gives a specialization preorder  $\lesssim$  on  $D$ , such that each closed set  $\overline{X} \subseteq D$  is downward-closed w.r.t.  $\lesssim$ . Applying these facts to reducibility candidates, we get that every candidate is downward-closed w.r.t. the specialization preorder issued from the closure operator  $\mathcal{CR}(-)$ .

Moreover, we discuss necessary and sufficient conditions for a term to belong to a reducibility candidate. We get characterizations of the membership of a term to a candidate based on the notion of values (i.e. non-neutral terms, see Def. 4.1.4). This is in fact the meaning of the neutral term property (clause (CR1)): given a set  $X \subseteq \mathcal{SN}_{\mathcal{R}}$  which is stable by reduction, its closure  $\mathcal{CR}(X)$  only adds neutral terms.

Finally, we refine both that characterization and the specialization preorder of  $\mathcal{CR}(-)$  in the main result of this section: reducibility candidates are downward-closed w.r.t. the *weak observational preorder*  $\lesssim_{\mathcal{V}} \subseteq \mathcal{SN}_{\mathcal{R}} \times \mathcal{SN}_{\mathcal{R}}$  defined as

$$t \lesssim_{\mathcal{V}} u \quad \text{if and only if} \quad \forall v \in \mathcal{V}. \quad t \rightarrow_{\mathcal{R}}^* v \implies u \rightarrow_{\mathcal{R}}^* v .$$

This section generalizes to reducibility candidates  $\mathcal{CR}_{\mathcal{RE}}$  results published in [Rib07b] where the preorder  $\lesssim_{\mathcal{V}}$  has been first presented.

**Closure operators on powersets.** Given a closure operator  $\overline{(-)} : \mathcal{P}(D) \mapsto \mathcal{P}(D)$ , we let

$$\overline{\mathcal{P}(D)} =_{\text{def}} \{ \overline{X} \mid X \subseteq D \} .$$

We first look at the partition of a closed set  $X \in \overline{\mathcal{P}(D)}$  according to the least closed sets containing each of its elements. Given  $d \in D$ , we write  $\overline{d}$  for  $\overline{\{t\}}$ . The following proposition says that given  $X \in \overline{\mathcal{P}(D)}$ , the set of  $\overline{d}$  for  $d \in X$  is a basis of  $X$ .

**Proposition 4.5.1** *Given a closure operator  $\overline{(-)} : \mathcal{P}(D) \mapsto \mathcal{P}(D)$ , for all  $X \subseteq D$  we have*

$$\overline{X} = \bigcup \{\overline{d} \mid d \in X\} .$$

PROOF. By extensivity, we have  $d \in \overline{d}$  for all  $d \in D$ , hence  $\overline{X} \subseteq \bigcup \{\overline{d} \mid d \in X\}$ . Conversely, if  $d \in \overline{X}$  then  $\overline{d} \subseteq \overline{X}$  by extensivity and idempotency. It follows that  $\bigcup \{\overline{d} \mid d \in X\} \subseteq \overline{X}$ .  $\square$

We can apply Prop. 4.5.1 to reducibility candidates, since they are defined by the closure operator  $\mathcal{CR}(\_)$ . This will be useful in Sect. 6 to study the stability by union of reducibility candidates. Given  $t \in \mathcal{SN}_R$ , we write  $\mathcal{CR}(t)$  for  $\mathcal{CR}(\{t\})$ .

**Corollary 4.5.2** *For all  $C \in \mathcal{CR}_{RE}$  we have*

$$C = \bigcup \{\mathcal{CR}(t) \mid t \in C\} .$$

PROOF. By Prop. 4.5.1 and Lem. 4.3.4.  $\square$

Our second point is that a closure operator on  $\mathcal{P}(D)$  gives rise to a preorder, that we see as an observational preorder. When  $\overline{\mathcal{P}(D)}$  is a topology, it corresponds to the usual specialization preorder of  $\overline{\mathcal{P}(D)}$  [AC98].

**Definition 4.5.3 (Specialization Preorder)** *The specialization preorder  $\lesssim \subseteq D \times D$  of a closure operator  $\overline{(-)} : \mathcal{P}(D) \mapsto \mathcal{P}(D)$  is defined as*

$$d \lesssim e \quad \text{if and only if} \quad \overline{d} \subseteq \overline{e} .$$

Note that we have  $d \in \overline{e}$  if and only if  $d \lesssim e$ . The intuition is that if  $d \in \overline{e}$ , then every observation made on  $e$  is also made on  $d$ . Hence  $d \lesssim e$  means that  $d$  is more precisely characterized by its observations than  $e$ .

**Proposition 4.5.4** *For all  $d, e \in D$ , we have  $d \in \overline{e}$  if and only if  $d \lesssim e$ .*

PROOF. Indeed, if  $d \lesssim e$ , then  $d \in \overline{d} \subseteq \overline{e}$ . Conversely, if  $d \in \overline{e}$  we have  $\overline{d} \subseteq \overline{e}$  by extensivity and idempotency of  $\overline{(-)}$ .  $\square$

Closed sets are downward-closed w.r.t.  $\lesssim$ . This means that if  $d \in X$  for some  $X \in \overline{\mathcal{P}(D)}$ , then every  $e$  which is more precisely characterized than  $d$  by its observations belongs to  $X$ .

**Proposition 4.5.5** *Every  $X \in \overline{\mathcal{P}(D)}$  is downward-closed w.r.t.  $\lesssim$ .*

PROOF. If  $e \in X$  and  $d \lesssim e$ , then  $d \in \overline{d} \subseteq \overline{e} \subseteq X = X$ .  $\square$

Hence reducibility candidates come with a specialization preorder, w.r.t. which each candidate is downward-closed.

**The values of reducibility candidates.** The goal of this paragraph is to obtain the material needed to get an interesting characterization of the specialization preorder of reducibility candidates. To this end, we discuss necessary and sufficient conditions for a term to belong to a reducibility candidate. The key point is that reducibility candidates are in some sense characterized by their values. We assume given a rewrite relation  $\rightarrow_R$  and a set  $\mathcal{E}$  of elimination contexts for  $\rightarrow_R$ .

**Definition 4.5.6** Given  $X \subseteq \Lambda(\Sigma)$ , the set  $\mathcal{V}(X)$  of values of  $X$  is defined as

$$\mathcal{V}(X) \quad =_{\text{def}} \quad \{v \in \mathcal{V} \mid v \in (X)_{\mathbf{R}}^*\} .$$

It is clear that  $\mathcal{V}(X) \subseteq \mathcal{SN}_{\mathbf{R}}$  whenever  $X \subseteq \mathcal{SN}_{\mathbf{R}}$ . Given  $t \in \Lambda(\Sigma)$ , we denote  $\mathcal{V}(\{t\})$  by  $\mathcal{V}(t)$ . We thus have  $\mathcal{V}(t) = \{v \in \mathcal{V} \mid t \rightarrow_{\mathbf{R}}^* v\}$ .

We first show that in some sense, a reducibility candidate is characterized by its values.

**Lemma 4.5.7 (First Characterization)** Given  $C \in \mathcal{CR}_{\mathcal{RE}}$  and  $t \in \mathcal{SN}_{\mathbf{R}}$ , we have

$$t \in C \quad \text{if and only if} \quad \mathcal{V}(t) \subseteq C .$$

PROOF. Since  $\mathcal{V}(t) \subseteq (t)_{\mathbf{R}}^*$ , by (CR0) it is clear that  $t \in C$  implies  $\mathcal{V}(t) \subseteq C$ .

For the converse, we reason by induction on  $t \in \mathcal{SN}_{\mathbf{R}}$ . So, let  $t \in \mathcal{SN}_{\mathbf{R}}$  such that  $\mathcal{V}(t) \subseteq C$ . If  $t$  is itself a value, then  $t \in \mathcal{V}(t)$  and we are done. Otherwise,  $t$  is neutral and by (CR1) it is sufficient to show that  $(t)_{\mathbf{R}} \subseteq C$ . But for all  $u \in (t)_{\mathbf{R}}$ , we have  $\mathcal{V}(u) \subseteq \mathcal{V}(t) \subseteq C$ , hence  $u \in C$  by induction hypothesis. It follows that  $t \in C$ .  $\square$

We thus have a first simple characterization of the membership of a term to a reducibility candidate. Note that in particular, if  $t \in \mathcal{HN}$ , we have  $\mathcal{V}(t) = \emptyset \subseteq C$  hence  $t \in C$  for all  $C \in \mathcal{CR}$ : this gives us a second proof that  $\mathcal{HN}$  is the least element of  $\mathcal{CR}$ .

The next step is to refine Lem. 4.5.7 into the following: given  $t \in \mathcal{SN}_{\mathbf{R}}$  and  $X \subseteq \mathcal{SN}_{\mathbf{R}}$ , we have

$$t \in \mathcal{CR}(X) \quad \text{if and only if} \quad \mathcal{V}(t) \subseteq \mathcal{V}(X) .$$

While Lem. 4.5.7 tells us that  $\mathcal{CR}(X)$  is characterized by its values  $\mathcal{V}(\mathcal{CR}(X))$ , this second characterization relies on the fact that the values of  $\mathcal{CR}(X)$  are exactly those of  $X$ . This is a consequence of the following proposition.

**Proposition 4.5.8** Given  $X \subseteq \mathcal{SN}_{\mathbf{R}}$  and  $t \in \mathcal{SN}_{\mathbf{R}}$ , we have

$$t \in \mathcal{CR}(X) \quad \text{if and only if} \quad [t \in (X)_{\mathbf{R}}^* \quad \text{or} \quad (t \in \mathcal{N}_{\mathcal{RE}} \quad \text{and} \quad (t)_{\mathbf{R}} \subseteq \mathcal{CR}(X))] .$$

PROOF. The "if" direction directly follows from the definition of  $\mathcal{CR}(\_)$  (Def. 4.3.3).

For "only if" direction, let  $\mathfrak{a} \in \mathfrak{D}$  be the least ordinal such that  $t \in \mathcal{CR}_{\mathfrak{a}}(X)$ . There are two cases: either  $\mathfrak{a} = 0$  and  $t \in (X)_{\mathbf{R}}^*$ , or  $\mathfrak{a} = \mathfrak{b} + 1$  and  $t \in \mathcal{N}_{\mathcal{RE}}$  with  $(t)_{\mathbf{R}} \subseteq \mathcal{CR}_{\mathfrak{b}}(X)$ .  $\square$

**Lemma 4.5.9 (Second Characterization)** Given  $X \subseteq \mathcal{SN}_{\mathbf{R}}$  and  $t \in \mathcal{SN}_{\mathbf{R}}$ , we have

$$t \in \mathcal{CR}(X) \quad \text{if and only if} \quad \mathcal{V}(t) \subseteq \mathcal{V}(X) .$$

PROOF. It follows from Prop. 4.5.8 that  $\mathcal{V}(\mathcal{CR}(X)) = \mathcal{V}(X)$ , and we conclude by Lem. 4.5.7.  $\square$

**Observational reducibility.** We can now give a direct definition of the specialization preorder of reducibility candidates. It follows from Lem. 4.5.9, which gives that  $t \in \mathcal{CR}(u)$  if and only if  $\mathcal{V}(t) \subseteq \mathcal{V}(u)$ , that is, if and only if every value of  $t$  is a value of  $u$ . We can formulate this with a *weak observational preorder*  $\sqsubseteq_{\mathcal{V}}$  on terms.

**Definition 4.5.10**

(i) We define  $\sqsubseteq_{\mathcal{V}} \subseteq \Lambda(\Sigma) \times \Lambda(\Sigma)$  as

$$t \sqsubseteq_{\mathcal{V}} u \quad \text{if and only if} \quad \mathcal{V}(t) \subseteq \mathcal{V}(u) .$$

(ii) We let  $\lesssim_{\mathcal{V}}$  be the restriction of  $\sqsubseteq_{\mathcal{V}}$  to  $\mathcal{SN}_{\mathcal{R}} \times \mathcal{SN}_{\mathcal{R}}$ , that is

$$t \lesssim_{\mathcal{V}} u \quad \text{if and only if} \quad (t, u \in \mathcal{SN}_{\mathcal{R}} \quad \text{and} \quad t \sqsubseteq_{\mathcal{V}} u) .$$

It is  $\lesssim_{\mathcal{V}}$ , the restriction of  $\sqsubseteq_{\mathcal{V}}$  to strongly normalizing terms, which is the specialization preorder of  $\mathcal{CR}(\cdot)$ . Before showing this in Lem. 4.5.13 below, let us make a few remarks.

First, it is clear that  $t \rightarrow_{\mathcal{R}} u$  implies  $u \sqsubseteq_{\mathcal{V}} t$  and moreover  $u \lesssim_{\mathcal{V}} t$  if  $t \in \mathcal{SN}_{\mathcal{R}}$ . Furthermore,  $\sqsubseteq_{\mathcal{V}}$  is compatible with elimination contexts.

**Proposition 4.5.11** *If  $t \sqsubseteq_{\mathcal{V}} u$ , then for all  $E[\ ] \in \mathcal{E}$  we have  $E[t] \sqsubseteq_{\mathcal{V}} E[u]$ .*

PROOF. For all  $n \in \mathbb{N}$ , all  $v \in \mathcal{V}$  and all  $E \in \mathcal{E}$ , we show that  $E[t] \rightarrow_{\mathcal{R}}^n v$  implies  $E[u] \rightarrow_{\mathcal{R}}^* v$  whenever  $t \sqsubseteq_{\mathcal{V}} u$ . We reason by induction on  $n$ .

If  $t$  is not neutral, then  $u \rightarrow_{\mathcal{R}}^* t$  since  $t \sqsubseteq_{\mathcal{V}} u$ . It follows that  $E[u] \rightarrow_{\mathcal{R}}^* E[t]$ , hence that  $E[t] \sqsubseteq_{\mathcal{V}} E[u]$ . Otherwise,  $t$  is neutral and we have  $n \geq 1$ . So, assume that  $E[t] \rightarrow_{\mathcal{R}} w \rightarrow_{\mathcal{R}}^n v$ . Since  $t$  is neutral, we have  $w = E'[t']$  with  $(E[\ ], t) \rightarrow_{\mathcal{R}} (E[\ ], t')$ , and there are two cases.

- If  $E[\ ] \rightarrow_{\mathcal{R}} E'[\ ]$  and  $t = t'$ , then since  $\mathcal{E}$  is stable by  $\rightarrow_{\mathcal{R}}$ , by induction hypothesis on  $n$  we have  $E'[u] \rightarrow_{\mathcal{R}}^* v$ , hence  $E[u] \rightarrow_{\mathcal{R}}^* v$ .
- Otherwise, we have  $t \rightarrow_{\mathcal{R}} t'$  and  $E'[\ ] = E[\ ]$ . Since  $t' \sqsubseteq_{\mathcal{V}} t$  and  $t \sqsubseteq_{\mathcal{V}} u$ , we have  $t' \sqsubseteq_{\mathcal{V}} u$  hence  $E[u] \rightarrow_{\mathcal{R}}^* v$  by induction hypothesis.  $\square$

Note that Prop. 4.5.11 implies that if  $[\ ] \in \mathcal{E}$ , then for all  $t, u \in \Lambda(\Sigma)$ ,

$$t \sqsubseteq_{\mathcal{V}} u \quad \text{if and only if} \quad \forall E[\ ] \in \mathcal{E}. \quad E[t] \sqsubseteq_{\mathcal{V}} E[u] .$$

**Remark 4.5.12** *Observational preorders were introduced to characterize behavioral equivalence: two pieces of programs are observationally equivalent iff when plugged in a program context, the obtained programs both diverges or evaluates to the same value.*

Usually, contexts are arbitrary terms  $C[\ ]$ , possibly under a binder. With closed terms, thanks to Milner's Context Lemma, this is equivalent to observation in applicative contexts (our elimination contexts  $\mathcal{E}_{\Rightarrow}$ ). Of course, this fails for open terms. See [AC98] for a presentation and references on the subject.

In the pure  $\lambda$ -calculus, closed values are abstractions, hence closed non-neutral terms correspond to the usual notion of value. Moreover, we have  $t \sqsubseteq_{\mathcal{V}} u$  iff for all  $E[\ ]$  we have  $E[t] \sqsubseteq_{\mathcal{V}} E[u]$ . Thus, with  $\sqsubseteq_{\mathcal{V}}$  we observe the reduction to values of open terms plugged in elimination contexts. Hence the name weak observational preorder.

We now show that  $\lesssim_{\mathcal{V}}$  is indeed the specialization preorder of  $\mathcal{CR}(\cdot)$ , in the sense of Def. 4.5.3.

**Lemma 4.5.13**  *$\lesssim_{\mathcal{V}}$  is the specialization preorder of  $\mathcal{CR}(\cdot)$ .*

PROOF. By Prop. 4.5.4 it is sufficient to show that  $t \lesssim_{\mathcal{V}} u$  if and only if  $t \in \mathcal{CR}(u)$ . But given  $t, u \in \mathcal{SN}_{\mathcal{R}}$ , by Lem. 4.5.9 we have  $t \in \mathcal{CR}(u)$  if and only if  $\mathcal{V}(t) \subseteq \mathcal{V}(u)$ , that is, if and only if  $t \lesssim_{\mathcal{V}} u$ .  $\square$

Note that Lem. 4.5.13 implies, given  $t \in \mathcal{SN}_{\mathcal{R}}$ , that  $\mathcal{CR}(t)$  is the the initial segment of  $(\mathcal{SN}_{\mathcal{R}}, \lesssim_{\mathcal{V}})$  whose maximal element is  $t$ , that is  $\mathcal{CR}(t) = \{u \mid u \lesssim_{\mathcal{V}} t\}$ .

We then arrive at our third and main characterization of the membership of a term to a candidate. It says that reducibility candidates are downward-closed w.r.t.  $\lesssim_{\mathcal{V}}$ . This follows from the combination of Cor. 4.5.2, which gives the topological structure of reducibility candidates as defined by a closure operator, and of Lem. 4.5.13, which characterizes the specialization preorder of reducibility candidates in terms of values.

**Theorem 4.5.14** *Reducibility candidates are downward-closed w.r.t.  $\lesssim_V$ .*

Our first characterization (Lem. 4.5.7) says that the membership of a strongly normalizing term to a candidate is entirely determined by the membership of its values to that candidate. Now, Thm 4.5.14 says that given  $t \in \mathcal{N}_{\mathcal{R}\mathcal{E}} \cap \mathcal{SN}_{\mathcal{R}}$ , a term  $u$  such that  $t \lesssim_V u$  captures all the relevant information on  $t$  w.r.t. reducibility: if  $u \in C$  for some  $C \in \mathcal{CR}_{\mathcal{R}\mathcal{E}}$ , then  $t \in C$ . This result will be strengthened in Cor. 7.1.1, after having studied and compared the stability by union of reducibility candidates and the closure by union of biorthogonals.

## 5 Biorthogonals

We now present a third reducibility family. This family is defined using an orthogonality relation between terms and elimination contexts. The idea underlying this interpretation comes from linear logic [Gir87]. Concerning reducibility, biorthogonals were introduced to take into account calculi for polarized classical logic: they have been used to show the strong normalization of the second order  $\lambda\mu$ -calculus [Par97], and they are the basis of Krivine's realizability for classical logic [DK00, Kri04]. Biorthogonals have also proved to be useful to show subtle computational properties of functional programming languages [Pit00, VM04, Vou04, MV05].

Let us try to present the main ideas in the case of the simply typed  $\lambda$ -calculus. Consider the interpretation of simple types given in Def. 3.2.2:

$$\begin{aligned} \llbracket B \rrbracket &= \mathcal{SN}_\beta & \text{if } B \in \mathcal{B} \\ \llbracket U \Rightarrow T \rrbracket &= \llbracket U \rrbracket \Rightarrow \llbracket T \rrbracket . \end{aligned}$$

Since any simple type  $T \in \mathcal{T}_{\Rightarrow}(\mathcal{B})$  can be written  $T = T_1 \Rightarrow \dots \Rightarrow T_n \Rightarrow B$ , its interpretation can be written  $\llbracket T_1 \rrbracket \Rightarrow \dots \llbracket T_n \rrbracket \Rightarrow \mathcal{SN}_\beta$ , that is

$$\llbracket T \rrbracket = \{ t \mid \forall u_1 \in \llbracket T_1 \rrbracket, \dots, u_n \in \llbracket T_n \rrbracket. \ t u_1 \dots u_n \in \mathcal{SN}_\beta \} .$$

It follows that if we map  $T$  to the set of elimination contexts

$$\llbracket T \rrbracket^\perp \stackrel{\text{def}}{=} \{ [ ] u_1 \dots u_n \mid u_1 \in \llbracket T_1 \rrbracket, \dots, u_n \in \llbracket T_n \rrbracket \} ,$$

then we have

$$\llbracket T \rrbracket = \left\{ t \mid \forall E[ ] . \ E[ ] \in \llbracket T \rrbracket^\perp \implies E[t] \in \mathcal{SN}_\beta \right\} .$$

Moreover, if we let  $t \perp\!\!\!\perp E[ ]$  when  $E[t] \in \mathcal{SN}_\beta$ , we obtain

$$\llbracket T \rrbracket = \left\{ t \mid \forall E[ ] . \ E[ ] \in \llbracket T \rrbracket^\perp \implies t \perp\!\!\!\perp E[ ] \right\} . \quad (26)$$

This construction is a form of *orthogonality*.

### 5.1 Orthogonality as a Galois Connection

We first present orthogonality as a Galois connection arising from a binary relation  $\perp\!\!\!\perp$  between two sets  $\mathcal{A}$  and  $\Pi$ . Results presented in this section are well-known, see e.g. [VM04, Vou04]. We give the proofs for the sake of completeness and because some of them are important in Sect. 6.4.

**Definition 5.1.1 (Orthogonality)** *Let  $\mathcal{A}$  and  $\Pi$  be two sets, and let  $\perp\!\!\!\perp \subseteq \mathcal{A} \times \Pi$ .*

— Given  $A \subseteq \mathcal{A}$ , the orthogonal of  $A$  w.r.t.  $\perp$  is

$$A^\perp =_{\text{def}} \{\pi \in \Pi \mid \forall \mathbf{a}. \mathbf{a} \in A \implies \mathbf{a} \perp \pi\}.$$

— Symmetrically, the orthogonal of  $P \subseteq \Pi$  w.r.t.  $\perp$  is

$$P^\perp =_{\text{def}} \{\mathbf{a} \in \mathcal{A} \mid \forall \pi. \pi \in P \implies \mathbf{a} \perp \pi\}.$$

The orthogonality operators  $(\_)^\perp : \mathcal{P}(\mathcal{A}) \mapsto \mathcal{P}(\Pi)$  and  $(\_)^\perp : \mathcal{P}(\Pi) \mapsto \mathcal{P}(\mathcal{A})$  form a Galois connection between the complete lattices  $(\mathcal{P}(\mathcal{A}), \subseteq)$  and  $(\mathcal{P}(\Pi), \supseteq)$ .

**Proposition 5.1.2 (Adjunction)** *For all  $A \subseteq \mathcal{A}$  and all  $P \subseteq \Pi$ , we have*

$$A \subseteq P^\perp \text{ if and only if } P \subseteq A^\perp.$$

PROOF. We only show one direction, the other one being symmetric. Assume that  $A \subseteq P^\perp$  and let us show that  $P \subseteq A^\perp$ . By assumption, for all  $\mathbf{a} \in A$ , we have  $\mathbf{a} \perp \pi$  for all  $\pi \in P$ . Hence for all  $\pi \in P$ , we have  $\mathbf{a} \perp \pi$  for all  $\mathbf{a} \in A$ . It follows that  $P \subseteq A^\perp$ .  $\square$

Galois connections induce closure operators.

**Proposition 5.1.3** *The maps  $(\_)^\perp{}^\perp : \mathcal{P}(\mathcal{A}) \mapsto \mathcal{P}(\mathcal{A})$  and  $(\_)^\perp{}^\perp : \mathcal{P}(\Pi) \mapsto \mathcal{P}(\Pi)$  are closure operators.*

PROOF. We only consider the case of  $(\_)^\perp{}^\perp : \mathcal{P}(\mathcal{A}) \mapsto \mathcal{P}(\mathcal{A})$ , because the other one is symmetric.

— Extensivity. For all  $X \subseteq \mathcal{A}$  we have  $X^\perp \subseteq X^\perp$ , hence  $X \subseteq X^\perp{}^\perp$  by Prop. 5.1.2.

We deduce from this property that  $(\_)^\perp$  is anti-monotone:

$$X \subseteq Y \implies Y^\perp \subseteq X^\perp. \quad (27)$$

Indeed, if  $X \subseteq Y$ , then by extensivity we have  $X \subseteq Y^\perp{}^\perp$ , hence  $Y^\perp \subseteq X^\perp$  by Prop. 5.1.2.

— Monotonicity. If  $X \subseteq Y$  then by (27) we have  $Y^\perp \subseteq X^\perp$ , hence  $X^\perp{}^\perp \subseteq Y^\perp{}^\perp$  by (27) again.

— Idempotency. Because

$$X^\perp = X^\perp{}^\perp{}^\perp. \quad (28)$$

Indeed, we have  $X^\perp \subseteq X^\perp{}^\perp{}^\perp$  by extensivity and  $X^\perp{}^\perp{}^\perp \subseteq X^\perp$  by applying Prop. 5.1.2 to  $X \subseteq X^\perp{}^\perp{}^\perp$ , which is obtained by two applications of extensivity.  $\square$

**Definition 5.1.4 (Biorthogonals)**

- (i) A set  $A \subseteq \mathcal{A}$  (resp.  $P \subseteq \Pi$ ) is a biorthogonal if  $A = A^\perp{}^\perp$  (resp.  $P = P^\perp{}^\perp$ ).
- (ii) We denote by  $\mathcal{P}(\mathcal{A})^\perp{}^\perp$  (resp.  $\mathcal{P}(\Pi)^\perp{}^\perp$ ) the set of biorthogonals of  $\mathcal{P}(\mathcal{A})$  (resp.  $\mathcal{P}(\Pi)$ ), and by  $\mathcal{P}^*(\mathcal{A})^\perp{}^\perp$  (resp.  $\mathcal{P}^*(\Pi)^\perp{}^\perp$ ) the set of biorthogonals of non-empty subsets of  $\mathcal{P}(\mathcal{A})$  (resp.  $\mathcal{P}(\Pi)$ ).

**Proposition 5.1.5**  *$A \subseteq \mathcal{A}$  (resp.  $P \subseteq \Pi$ ) is a biorthogonal if and only if there exists  $X \subseteq \Pi$  (resp.  $X \subseteq \mathcal{A}$ ) such that  $A = X^\perp$  (resp.  $P = X^\perp$ ).*

PROOF. If  $A$  is a biorthogonal then by definition we have  $A = (A^\perp)^\perp$ . Conversely, if  $A = Y^\perp$  then  $A^\perp{}^\perp = Y^\perp{}^\perp{}^\perp$ , hence  $A^\perp{}^\perp = Y^\perp = A$  by (28).  $\square$

## 5.2 Biorthogonals for Reducibility

Let us now present how these ideas can be applied to reducibility. We define type interpretations based on biorthogonality relying on the relation suggested by (26). We stay in the same framework as for reducibility candidates in Sect. 4. Therefore, we consider a rewrite relation  $\rightarrow_R$  on  $\Lambda(\Sigma)$  and a set  $\mathcal{E}$  of elimination contexts for  $\rightarrow_R$  in the sense of Def. 4.2.1.

**Definition 5.2.1** *We define  $\perp\!\!\!\perp \subseteq \Lambda(\Sigma) \times \mathcal{E}$  as*

$$t \perp\!\!\!\perp E[\ ] \quad \text{if and only if} \quad E[t] \in \mathcal{SN}_R.$$

The non-empty subsets of  $\mathcal{SN}_R$  which are biorthogonals w.r.t.  $\perp\!\!\!\perp$  are reducibility candidates in the sense of Def. 4.2.2. Note that non-emptiness is mandatory. Similarly to happen with  $\_ \Rightarrow \_$  in property (8) (on page 11 in Sect. 3.2), the orthogonal in  $\mathcal{E}$  of  $\emptyset \in \mathcal{P}(\Lambda(\Sigma))$  is  $\mathcal{E}$ :

$$\emptyset^{\perp\!\!\!\perp} = \{E[\ ] \in \mathcal{E} \mid \forall t. \ t \in \emptyset \implies t \perp\!\!\!\perp E[\ ]\} = \mathcal{E}.$$

Hence the biorthogonal of  $\emptyset \in \mathcal{P}(\Lambda(\Sigma))$  is  $\Lambda(\Sigma)$  itself.

**Lemma 5.2.2**  $\mathcal{P}^*(\mathcal{SN}_R)^{\perp\!\!\!\perp} \subseteq \mathcal{CR}_{\mathcal{E}}$ .

PROOF. Let  $C \in \mathcal{P}^*(\mathcal{SN}_R)^{\perp\!\!\!\perp}$ . We check that  $C$  satisfies the clauses (CR0) and (CR1).

(CR0) If  $t \in C$  and  $t \rightarrow_R u$ , then for all context  $E[\ ] \in C^{\perp\!\!\!\perp}$  we have  $E[t] \rightarrow_R E[u]$ , hence  $E[u] \in \mathcal{SN}_R$ .

(CR1) Let  $t \in \mathcal{N}_{\mathcal{R}\mathcal{E}}$  be such that  $(t)_R \subseteq C$ . We must show that  $E[t] \in \mathcal{SN}_R$  for all  $E[\ ] \in C^{\perp\!\!\!\perp}$ . Since  $C$  is a non-empty subset of  $\mathcal{SN}_R$ , we have  $C^{\perp\!\!\!\perp} \subseteq \mathcal{SN}_R$ . We reason by induction on  $E[\ ] \in \mathcal{SN}_R$ . Let  $v$  such that  $E[t] \rightarrow_R v$ . Since  $t$  is neutral, by definition we have  $v = E'[t']$  with  $(E[\ ], t) \rightarrow_R (E'[\ ], t')$ , and there are two cases:

- $v = E[u]$  with  $t \rightarrow_R u$ . In this case we have  $E[u]$  since  $u \in C$  and  $E[\ ] \in C^{\perp\!\!\!\perp}$ .
- $v = E'[t']$  with  $E[\ ] \rightarrow_R t'$ . We then have  $E'[\ ] \in C^{\perp\!\!\!\perp}$  because for all  $u \in C$ ,  $E[u] \rightarrow_R E'[u]$  and  $E[u] \in \mathcal{SN}_R$ , hence  $E'[u] \in \mathcal{SN}_R$ . It follows that  $E'[t] \in \mathcal{SN}_R$  by induction hypothesis.  $\square$

Therefore, in the  $\lambda$ -calculus with products, since  $\mathcal{E}_{\Rightarrow \times}$  is a set of elimination contexts for  $\rightarrow_{\beta\pi}$ , by Lem. 4.2.6 we have  $\mathcal{P}^*(\mathcal{SN}_{\beta\pi})^{\perp\!\!\!\perp} \subseteq \mathcal{SAT}_{\beta\pi}$ .

In order to get a valid and adequate interpretation, it remains to show that the function and product space constructors preserve biorthogonals. To this end, it is interesting to observe, in the case of the pure  $\lambda$ -calculus, how the type interpretation (26) goes through the structure of simple types. In the base case, we have  $\llbracket T \rrbracket = \mathcal{SN}_{\beta} = \{\llbracket \_ \rrbracket\}^{\perp\!\!\!\perp}$ . Consider now a type of the form  $U \Rightarrow T$ . The main idea is that for all  $t, u \in \Lambda(\Sigma)$  and all  $E[\ ] \in \mathcal{E}_{\Rightarrow}$ , we have

$$t u \perp\!\!\!\perp E[\ ] \quad \text{if and only if} \quad t \perp\!\!\!\perp E[\llbracket \_ \rrbracket u].$$

It follows that

$$\begin{aligned} \llbracket U \rrbracket \Rightarrow \llbracket T \rrbracket &= \{t \mid \forall u \in \llbracket U \rrbracket, \forall E[\ ] \in \llbracket T \rrbracket^{\perp\!\!\!\perp}. \ t u \perp\!\!\!\perp E[\ ]\} \\ &= \{t \mid \forall u \in \llbracket U \rrbracket, \forall E[\ ] \in \llbracket T \rrbracket^{\perp\!\!\!\perp}. \ t \perp\!\!\!\perp E[\llbracket \_ \rrbracket u]\} \\ &= \{E[\llbracket \_ \rrbracket u] \mid u \in \llbracket U \rrbracket \ \wedge \ E[\ ] \in \llbracket T \rrbracket^{\perp\!\!\!\perp}\}^{\perp\!\!\!\perp}. \end{aligned}$$



**Definition 5.2.3** If  $[ ]t \in \mathcal{E}$  for all  $t \in \Lambda(\Sigma)$ , then given  $A \subseteq \Lambda(\Sigma)$  and  $P \subseteq \mathcal{E}$ , we let

$$A \cdot P \stackrel{\text{def}}{=} \{E[ ]u \mid u \in A \wedge E[ ] \in P\}.$$

We now show that the function and product spaces constructor preserve biorthogonals.

**Proposition 5.2.4**

(i) If  $[ ]t \in \mathcal{E}$  for all  $t \in \Lambda(\Sigma)$ , then for all  $A, B \subseteq \mathcal{SN}_{\mathcal{R}}$  we have

$$A \Rightarrow B^{\perp\perp} = (A \cdot B^{\perp})^{\perp}.$$

(ii) If  $\pi_i[ ] \in \mathcal{E}$  for all  $i \in \{1, 2\}$ , then for all  $A_1, A_2 \subseteq \mathcal{SN}_{\mathcal{R}}$  we have

$$A_1^{\perp\perp} \times A_2^{\perp\perp} = \{E[\pi_1[ ]] \mid E[ ] \in A_1^{\perp}\}^{\perp} \cap \{E[\pi_2[ ]] \mid E[ ] \in A_2^{\perp}\}^{\perp}.$$

PROOF.

(i) We have  $t \in A \Rightarrow B^{\perp\perp}$  if and only if  $E[tu] \in \mathcal{SN}_{\mathcal{R}}$  for all  $u \in A$  and all  $E[ ] \in B^{\perp}$ .

(ii) We have  $t \in A_1^{\perp\perp} \times A_2^{\perp\perp}$  if and only if  $E[\pi_i t] \in \mathcal{SN}_{\mathcal{R}}$  for all  $i \in \{1, 2\}$  and all  $E[ ] \in A_i^{\perp}$ .  $\square$

According to Prop 5.1.5 and Prop. 5.1.3, it follows that  $A^{\perp\perp} \Rightarrow B^{\perp\perp}$  and  $A^{\perp\perp} \times B^{\perp\perp}$  are biorthogonals for all  $A, B \subseteq \mathcal{SN}_{\beta\pi}$ . If moreover  $A$  and  $B$  are not empty, then  $A^{\perp\perp}$  and  $B^{\perp\perp}$  are saturated sets, hence  $A^{\perp\perp} \Rightarrow B^{\perp\perp}, A^{\perp\perp} \times B^{\perp\perp} \in \mathcal{SAT}_{\beta\pi}$  by Prop. 3.3.3. Hence biorthogonals provide a valid and adequate type interpretation for the  $\lambda$ -calculus with products.

## 6 Stability by Union

A reducibility family  $\mathcal{Red}$  is *stable by union* if

$$\forall \mathcal{R}. \quad \emptyset \neq \mathcal{R} \subseteq \mathcal{Red} \implies \bigcup \mathcal{R} \in \mathcal{Red}.$$

The main question concerning stability by union is the following: given a rewrite relation  $\rightarrow_{\mathcal{R}}$ , does there exists a reducibility family  $\mathcal{Red}$  for  $\rightarrow_{\mathcal{R}}$  which is stable by union *and* leads to an adequate type interpretation ?

For the pure  $\lambda$ -calculus, it is well known that the answer is positive: Tait's saturated sets (presented in Sec. 2.2) are stable by union and lead to an adequate type interpretation. This has been exploited for instance in [Abe06a, Tat07].

But with rewriting, the question is more difficult. We have seen in Sec. 2.2 that because rewrite systems in general do not satisfy the weak standardization lemma (Lem 3.2.8), in general we need a reducibility family satisfying a clause like (19). But this is precisely what makes stability by union difficult. Assume given  $\mathcal{R} \subseteq \mathcal{Red}$  such that for all  $v \in (E[f(\vec{t})])_{\beta\mathcal{R}}$ , we have  $v \in \bigcup \mathcal{R}$ . Then, unless we find  $A \in \mathcal{R}$  such that  $(E[f(\vec{t})])_{\beta\mathcal{R}} \subseteq A$ , there is no reason to have  $E[f(\vec{t})] \in \bigcup \mathcal{R}$ .

By the way, using intersection and union types, we have shown in [Rib07c] that there are confluent rewrite systems for which every reducibility family that leads to an adequate type interpretation is not stable by union. However, there are cases in which we can obtain a reducibility family which is stable by union.

In this section, we study necessary and sufficient conditions for Girard's reducibility candidates to be stable by union; and a necessary and sufficient condition for the closure by union of biorthogonals to be reducibility candidates. This generalizes results of [Rib07b]. We will see that the second condition is strictly more general than the first one. Hence, reducibility candidates strictly contain a reducibility family which can be stable by union even when they are themselves not stable by union.



## 6.1 Closure by Union of a Closure Operator

When trying to get a stable by union reducibility family, it is convenient to think on what happens at the level of closure operators. Therefore, we begin by looking at the shape of the closure by union of a closure operator  $\overline{(-)}$  on  $\mathcal{P}(D)$ , where  $D$  is some arbitrary set.

Similarly as with biorthogonals in Sect. 5, given a closure operator  $\overline{(-)}$  on  $\mathcal{P}(D)$ , we denote by  $\overline{\mathcal{P}^*(D)}$  the set of closures of elements of  $\mathcal{P}(D) \setminus \{\emptyset\}$ :

$$\overline{\mathcal{P}^*(D)} =_{\text{def}} \{ \overline{X} \mid \emptyset \neq X \subseteq D \} .$$

In Sect. 4.5, we have seen that a closure operator  $\overline{(-)} : \mathcal{P}(D) \mapsto \mathcal{P}(D)$  gives a specialization preorder  $\lesssim \subseteq D \times D$  such that every closed set  $\overline{X} \in \overline{\mathcal{P}(D)}$  is downward-closed w.r.t.  $\lesssim$ . The closure by union of  $\overline{\mathcal{P}^*(D)}$  is the set  $\mathcal{O}$  of all non-empty subsets of  $D$  which are downward-closed w.r.t.  $\lesssim$ .

**Definition 6.1.1** *Let  $\mathcal{O}$  be the set of non-empty  $X \subseteq D$  which are downward-closed w.r.t.  $\lesssim$ .*

To show that  $\mathcal{O}$  is the closure by union of  $\overline{\mathcal{P}^*(D)}$ , it is convenient to use the following property on the structure of elements of  $\mathcal{O}$  (see also Prop. 4.5.1).

**Proposition 6.1.2**  *$X \in \mathcal{O}$  if and only if  $X = \bigcup \{ \overline{d} \mid d \in X \}$ .*

PROOF. We begin by the "only if" direction. Given  $X \in \mathcal{O}$ , if  $d \in X$ , then  $d \in \overline{d}$ . Conversely, if  $d \in \overline{e}$  for some  $e \in X$ , then  $d \lesssim e$  hence  $d \in X$  since  $X$  is downward-closed.

For the "if" direction, if  $X = \bigcup \{ \overline{d} \mid d \in X \}$ , then  $X$  is downward-closed since every  $\overline{d}$  is downward-closed by Prop. 4.5.5.  $\square$

We now show that  $\mathcal{O}$  is the closure by union of  $\overline{\mathcal{P}^*(D)}$ .

**Proposition 6.1.3** *Given a closure operator  $\overline{(-)} : \mathcal{P}(D) \mapsto \mathcal{P}(D)$ ,  $\mathcal{O}$  is the least set such that*

$$\overline{\mathcal{P}^*(D)} \subseteq \mathcal{O} \quad \text{and} \quad \left( \emptyset \neq \mathcal{C} \subseteq \mathcal{O} \implies \bigcup \mathcal{C}, \bigcap \mathcal{C} \in \mathcal{O} \right) .$$

PROOF. Stability by intersection and union are both trivial.

We show that  $\overline{\mathcal{P}^*(D)} \subseteq \mathcal{O}$ . Given  $X \in \overline{\mathcal{P}^*(D)}$  and  $d, e$  such that  $d \lesssim e$  and  $e \in X$ , we have  $\overline{d} \subseteq \overline{e} \subseteq X$ , hence  $d \in X$  since  $d \in \overline{d}$ .

Let  $\Omega$  be a set containing  $\overline{\mathcal{P}^*(D)}$  and which is stable by intersection and union. We have to show that  $\mathcal{O} \subseteq \Omega$ . If  $X \in \mathcal{O}$ , then  $X = \bigcup \{ \overline{d} \mid d \in X \}$  by the above observation. But  $\{ \overline{d} \mid d \in X \} \subseteq \Omega$  since for all  $d \in X$  we have  $\overline{d} \in \overline{\mathcal{P}^*(D)}$ . It follows that  $\bigcup \{ \overline{d} \mid d \in X \} \in \Omega$ .  $\square$

Recall that by Lem. 2.2.6, the set  $\overline{\mathcal{P}^*(D)}$  is itself closed by non-empty intersections. Hence, Prop. 6.1.3 implies that  $\overline{\mathcal{P}^*(D)}$  is closed by union if and only if  $\overline{\mathcal{P}^*(D)} = \mathcal{O}$ .

## 6.2 Saturated Sets

We begin by considering the case of saturated sets. For the pure  $\lambda$ -calculus and some of its extensions, such as the  $\lambda$ -calculus with products, stability by union of saturated sets is straightforward and well-known, see for instance [Wer94, Abe06b, Tat07]. In general, this is not the case of the combination of  $\lambda$ -calculus with rewriting. We will see in Sect. 6.4 that Ex. 3.5.1, which prevented us from uniformly defining saturated sets for rewriting, hides in fact a problem of stability by union.

In this section, we briefly review why saturated sets  $\mathcal{SAT}_{\beta\pi}$  for the  $\lambda$ -calculus with products are stable by union. Recall that  $\mathcal{SAT}_{\beta\pi}$  is defined in Def. 3.3.1. Moreover, as seen in Sect. 3.4, these saturated sets are defined by the closure operator  $\mathcal{SAT}(-) : \mathcal{P}(\mathcal{SN}_{\beta\pi}) \mapsto \mathcal{P}(\mathcal{SN}_{\beta\pi})$ . By Prop. 4.5.1, every  $S \in \mathcal{SAT}_{\beta\pi}$  satisfies

$$S = \bigcup \{ \mathcal{SAT}(t) \mid t \in S \} . \quad (29)$$

Now, it follows from Prop. 6.1.2 and Prop. 6.1.3 that  $\mathcal{SAT}_{\beta\pi}$  is stable by union if and only if all non-empty  $S \subseteq \mathcal{SN}_{\beta\pi}$  satisfying (29) is a saturated set. This can be easily checked.

**Theorem 6.2.1** *The set  $\mathcal{SAT}_{\beta\pi}$  is closed by union.*

PROOF. Let  $S \subseteq \mathcal{SN}_{\beta\pi}$  satisfying (29).

The clause ( $\mathcal{SAT}1$ ) is obvious: if  $E[\ ] \in \mathcal{E}_{\Rightarrow \times} \cap \mathcal{SN}_{\beta\pi}$  and  $x \in \mathcal{X}$  then  $E[x] \in \mathcal{SAT}(t)$  for all  $t \in S$ , hence  $E[x] \in S$ .

The clauses ( $\mathcal{SAT}2_\beta$ ) and ( $\mathcal{SAT}2_{\pi i}$ ) can be dealt with together. Let  $t \mapsto_{\beta\pi} u$  such that  $t \in \mathcal{SN}_{\beta\pi}$  and  $E[u] \in S$ . We have  $E[u] \in \mathcal{SAT}(v)$  for some  $v \in S$ , hence  $E[t] \in \mathcal{SAT}(v) \subseteq S$ .  $\square$

### 6.3 Reducibility Candidates

We now discuss the case of reducibility candidates. We begin by a general characterization of their stability by union which is based on the material of Sect. 4.5, and which generalizes results of [Rib07b]. We then apply it to show that for the pure  $\lambda$ -calculus, the  $\lambda$ -calculus with products, and the combination of  $\lambda$ -calculus with orthogonal constructor rewriting, reducibility candidates are stable by union. The proofs for the pure  $\lambda$ -calculus and the  $\lambda$ -calculus with products are adapted from [Rib07b]. For orthogonal rewriting, the proof is published in [Rib08].

**Stability by union.** Let  $\rightarrow_R$  be a rewrite relation on  $\Lambda(\Sigma)$  and  $\mathcal{E}$  be a set of elimination contexts for  $\rightarrow_R$ . In Sect. 4.5, we have seen that the specialization preorder of  $\mathcal{CR}(-)$  is the preorder  $\lesssim_{\mathcal{V}} \subseteq \mathcal{P}(\mathcal{SN}_R) \times \mathcal{P}(\mathcal{SN}_R)$  defined as

$$t \lesssim_{\mathcal{V}} u \quad \text{if and only if} \quad \forall v \in \mathcal{V}. \quad t \rightarrow^* v \implies u \rightarrow^* v .$$

Therefore, according to the discussion after Prop. 6.1.3,  $\mathcal{CR}$  is stable by union if and only if it is the set  $\mathcal{O}_{\mathcal{V}}$  of all non-empty subsets of  $\mathcal{SN}_R$  which are downward-closed w.r.t.  $\lesssim_{\mathcal{V}}$ .

**Definition 6.3.1** *We let  $\mathcal{O}_{\mathcal{V}}$  be the set of non-empty subsets of  $\mathcal{SN}_R$  which are downward-closed w.r.t.  $\lesssim_{\mathcal{V}}$ .*

There are rewrite relations with which reducibility candidates are not stable by union. This means that there are rewrite relations  $\rightarrow_R$  and elimination contexts  $\mathcal{E}_{\Rightarrow}$  such that  $\mathcal{CR}_{\mathcal{R}\mathcal{E}}$  is strictly included in  $\mathcal{O}_{\mathcal{V}}$ . Besides the examples of [Rib07c], for which there are no reducibility families which are stable by union, there are very simple confluent rewrite systems for which reducibility candidates are not stable by union.

**Example 6.3.2** *Consider the confluent system*

$$p \mapsto_R \lambda x.c_1 \qquad p \mapsto_R \lambda x.c_2 \qquad c_i \mapsto_R d .$$

Taking  $\mathcal{E}_{\Rightarrow}$  as elimination contexts, the values for  $\rightarrow_{\beta R}$  in  $\mathcal{E}_{\Rightarrow}$  are exactly the terms of the form  $\lambda x.t$ . Consider now the set

$$C \stackrel{\text{def}}{=} \mathcal{CR}(\lambda x.c_1) \cup \mathcal{CR}(\lambda x.c_2) .$$

Since the sets  $\mathcal{CR}(\lambda x.c_i)$  are downward-closed w.r.t.  $\lesssim_V$ , it is clear that  $C$  is downward-closed w.r.t.  $\lesssim_V$ . Hence, if  $\mathcal{CR}_{\beta\mathcal{R}}$  were stable by union, then we would have  $C \in \mathcal{CR}_{\beta\mathcal{R}}$ .

However, this is not the case. First, note that  $\lambda x.c_1$  and  $\lambda x.c_2$  are two distinct values. Hence they are not comparable w.r.t.  $\lesssim_V$ . Therefore, given  $i \in \{1, 2\}$ , we have  $\lambda x.c_{3-i} \notin \mathcal{CR}(\lambda x.c_i)$ . But since  $p \rightarrow_{\mathcal{R}} \lambda x.c_{3-i}$ , this implies that  $p \notin \mathcal{CR}(\lambda x.c_i)$ . It follows that  $p \notin C$ , while  $p$  is a neutral term such that  $(p)_{\beta\mathcal{R}} = \{\lambda x.c_1, \lambda x.c_2\} \subseteq C$ .

It is therefore interesting to see what can ensure  $\mathcal{O}_V \subseteq \mathcal{CR}_{\mathcal{RE}}$ . Given  $C \in \mathcal{O}_V$ , let us see under which conditions we can have  $C \in \mathcal{CR}_{\mathcal{RE}}$ . First, the clause  $(\mathcal{CR}0)$  is trivially satisfied: given  $t \in C$  and  $u \in (t)_{\mathcal{R}}$ , we have  $u \lesssim_V t$  hence  $u \in C$ .

Concerning the neutral term property (clause  $(\mathcal{CR}1)$ ), let  $t \in \mathcal{SN}_{\mathcal{R}} \cap \mathcal{N}_{\mathcal{RE}}$  be a reducible term such that  $(t)_{\mathcal{R}} \subseteq C$ . Since  $C$  is downward-closed w.r.t.  $\lesssim_V$ , the only general way to get  $t \in C$  is to ensure that there is  $u \in C$  such that  $t \lesssim_V u$ . Now, take

$$C(t) \stackrel{\text{def}}{=} \bigcup \{ \mathcal{CR}(u) \mid u \in (t)_{\mathcal{R}} \}.$$

In this case, we have  $t \in C(t)$  if and only if there is  $u \in (t)_{\mathcal{R}}$  such that  $t \lesssim_V u$ . In other words, every strongly normalizing reducible neutral term  $t$  must have a reduct  $u$  such that every value of  $t$  is a value of  $u$ . Such a  $u$  is a *strong principal reduct* of  $t$ .

**Definition 6.3.3 (Strong Principal Reduct)** *Given a reducible term  $t \in \mathcal{N}_{\mathcal{RE}} \cap \mathcal{SN}_{\mathcal{R}}$ , a term  $u \in (t)_{\mathcal{R}}$  such that  $t \lesssim_V u$  is a strong principal reduct of  $t$ <sup>1</sup>.*

*We say that  $\rightarrow_{\mathcal{R}}$  satisfies the strong principal reduct property or that  $\rightarrow_{\mathcal{R}}$  has strong principal reducts when every reducible strongly normalizing neutral term has a strong principal reduct.*

We now characterize the stability by union of reducibility candidates. In fact, ensuring that the set  $C(t) \in \mathcal{O}_V$  above is a reducibility candidate for all reducible  $t \in \mathcal{SN}_{\mathcal{R}} \cap \mathcal{N}_{\mathcal{RE}}$  amounts to ensuring that all  $C \in \mathcal{O}_V$  are reducibility candidates.

**Theorem 6.3.4 (Stability by Union of Reducibility Candidates)** *The following are equivalent:*

- (i)  $\mathcal{CR}_{\mathcal{RE}}$  is stable by union,
- (ii)  $\mathcal{CR}_{\mathcal{RE}} = \mathcal{O}_V$ ,
- (iii) every reducible strongly normalizable neutral term  $t$  has a strong principal reduct.

PROOF.

**(i)  $\implies$  (ii).** Let  $C \subseteq \mathcal{SN}_{\mathcal{R}}$  be a non-empty set downward-closed w.r.t.  $\lesssim_V$ . Since  $\mathcal{CR}(t)$  is downward closed w.r.t.  $\lesssim_V$  for all  $t \in \mathcal{SN}_{\mathcal{R}}$ , we have  $C = \bigcup \{ \mathcal{CR}(t) \mid t \in C \}$ . Hence  $C \in \mathcal{CR}$  because  $\mathcal{CR}$  is stable by union.

**(ii)  $\implies$  (iii).** Let  $t \in \mathcal{N} \cap \mathcal{SN}_{\mathcal{R}}$  be reducible. For all  $u \in (t)_{\mathcal{R}}$ , the set  $\mathcal{CR}(u)$  is non-empty and downward-closed w.r.t.  $\lesssim_V$ . Therefore, the set  $C$  of all  $v$  such that  $v \in \mathcal{CR}(u)$  for some  $u \in (t)_{\mathcal{R}}$  is non-empty and downward-closed w.r.t.  $\lesssim_V$ . It follows that  $C \in \mathcal{CR}$  and that  $t \in C$  since  $(t)_{\mathcal{R}} \subseteq C$ . Hence there is  $u \in (t)_{\mathcal{R}}$  such that  $t \lesssim_V u$ .

---

<sup>1</sup>Called "principal reduct" in [Rib07b].

(iii)  $\implies$  (i). Let  $\emptyset \neq \mathcal{C} \subseteq \mathcal{CR}$ . In order to show  $\bigcup \mathcal{C} \in \mathcal{CR}$ , the key-point is to show that if  $t \in \mathcal{N}$  is such that  $(t)_R \subseteq \bigcup \mathcal{C}$  then  $t \in \bigcup \mathcal{C}$ . If  $(t)_R = \emptyset$  then  $t \in C$  for all  $C \in \mathcal{C}$  and we are done. Otherwise, we have  $t \in \mathcal{SN}_R$  since  $(t)_R \subseteq \bigcup \mathcal{C} \subseteq \mathcal{SN}_R$ . Let  $u$  be a strong principal reduct of  $t$ . There is  $C \in \mathcal{C}$  such that  $u \in C$ , and since  $t \lesssim_{\mathcal{V}} u$  and  $C$  is downward-closed w.r.t.  $\lesssim_{\mathcal{V}}$ , we have  $t \in C$ , hence  $t \in \bigcup \mathcal{C}$ .  $\square$

This result deserves a few remarks. First, by Thm. 4.5.14, we always have  $\mathcal{CR}_{\mathcal{RE}} \subseteq \mathcal{O}_{\mathcal{V}}$ . Hence, Thm. 6.3.4 says that  $\mathcal{CR}_{\mathcal{RE}}$  is stable by union if and only if  $\mathcal{O}_{\mathcal{V}} \subseteq \mathcal{CR}_{\mathcal{RE}}$ .

The second and more important point is that Thm 6.3.4 gives an interesting information on the structure of Girard's reducibility candidates. It says that Girard's reducibility candidates are stable by union exactly when they are exactly the non-empty subsets of  $\mathcal{SN}_R$  which are downward-closed w.r.t.  $\lesssim_{\mathcal{V}}$ . This gives a nice "straight" structure to candidates, which is very simple compared to that appearing in their definition. Of course, this makes sense only if there are non-trivial rewrite relations for which this holds.

We now show that it is actually the case for the pure  $\lambda$ -calculus, the  $\lambda$ -calculus with products and the combination of  $\lambda$ -calculus with orthogonal constructor rewriting.

**Pure lambda-calculus.** We show that for the pure  $\lambda$ -calculus, reducibility candidates  $\mathcal{CR}_{\beta}$  are stable by union. This property has also been shown by M. Tatsuta<sup>2</sup>. Recall that in Ex. 4.2.3, we have defined  $\mathcal{CR}_{\beta}$  as the set of reducibility candidates for  $\rightarrow_{\beta}$  in the elimination contexts

$$E[\ ] \in \mathcal{E}_{\Rightarrow} ::= [\ ] \mid E[\ ] t.$$

Moreover, recall that the neutral terms for  $\rightarrow_{\beta}$  in  $\mathcal{E}_{\Rightarrow}$  are exactly the terms of the form

$$E[x] \quad \text{or} \quad E[(\lambda x.t)u] \quad \text{with} \quad E[\ ] \in \mathcal{E}_{\Rightarrow}.$$

By Thm. 6.3.4, the stability by union of  $\mathcal{CR}_{\beta}$  is equivalent to the strong principal reduct property, which holds in the case of the pure  $\lambda$ -calculus thanks to non-interaction (Lem. 3.2.7) and weak standardization (Lem. 3.2.8). This means that the stability by union of  $\mathcal{CR}_{\beta}$  relies on the same properties as those used in Sect. 3.2 to show that saturated sets define an adequate type interpretation. We further discuss this point in Sect. 7.2.

The strong principal reduct property follows from the fact that weak-head  $\beta$ -reduction gives strong principal reducts. We do not give the proof here, since it is subsumed by that of the  $\lambda$ -calculus with products (Prop. 6.3.11 below).

**Proposition 6.3.5** *For all  $E[\ ] \in \mathcal{E}_{\Rightarrow}$  we have  $E[(\lambda x.t)u] \sqsubseteq_{\mathcal{V}} E[t[u/x]]$ .*

**Lemma 6.3.6 (Strong Principal Reduct Property)** *In the pure  $\lambda$ -calculus, every reducible strongly normalizable neutral term  $t$  has a strong principal reduct.*

PROOF. Let  $t \in \mathcal{N}_{\beta} \cap \mathcal{SN}_{\beta}$  be reducible. There are two cases.

$t = E[x]$  with  $E[\ ] \in \mathcal{E}_{\Rightarrow} \cap \mathcal{SN}_{\beta}$ . We get  $t \in \mathcal{HN}_{\beta}$  by Prop. 4.4.4. Hence  $\mathcal{V}(t) = \emptyset$  and every  $u \in (t)_{\beta}$  is a strong principal reduct of  $t$ .

$t = E[(\lambda x.t_1)t_2]$ . By Prop. 6.3.5, and using that  $E[(\lambda x.t_1)t_2], E[t_1[t_2/x]] \in \mathcal{SN}_{\beta}$ .  $\square$

We thus obtain the stability by union of  $\mathcal{CR}_{\beta}$ .

---

<sup>2</sup>Private communication.

**Corollary 6.3.7**  $\mathcal{CR}_\beta$  is stable by union.

By Prop. 6.3.5, weak-head  $\beta$ -reduction gives strong principal reducts. It is interesting to note that the converse is false: as shown in the following example, there are neutral terms  $t$  with  $\mathcal{V}(t) \neq \emptyset$  that have strong principal reducts which are not weak-head reducts. The idea of this example comes from the notions of prime redex and of canonical form of [Reg94, DHR96].

**Example 6.3.8** Consider a strongly normalizing neutral term  $t$  of the form

$$(\lambda y_1. (\lambda y_2. \lambda x. v) u_2) u_1 \quad \text{where} \quad y_1, y_2 \notin v.$$

All the values of  $t$  are reducts of  $\lambda x. v$ . Of course, we can obtain  $\lambda x. v$  from  $t$  by first contracting the head-redex of  $t$ :

$$(\lambda y_1. (\lambda y_2. \lambda x. v) u_2) u_1 \mapsto_\beta (\lambda y_2. \lambda x. v) u_2 \mapsto_\beta \lambda x. v.$$

We can also obtain it by first contracting  $(\lambda y_2. \lambda x. v) u_2$ , which is not the weak-head redex of  $t$ :

$$(\lambda y_1. (\lambda y_2. \lambda x. v) u_2) u_1 \rightarrow_\beta (\lambda y_1. \lambda x. v) u_1 \mapsto_\beta \lambda x. v.$$

Hence,  $(\lambda y_1. \lambda x. v) u_1$  is a strong principal reduct of  $t$  but it is not a weak-head reduct of  $t$ .

**Lambda-calculus with products.** We now turn to the  $\lambda$ -calculus with products. Recall that in Ex. 4.2.3, we have defined  $\mathcal{CR}_{\beta\pi}$  as the set of reducibility candidates for  $\rightarrow_{\beta\pi}$  in elimination contexts

$$E[\ ] \in \mathcal{E}_{\Rightarrow \times} ::= [\ ] \mid E[\ ] t \mid \pi_1 E[\ ] \mid \pi_2 E[\ ].$$

As noted in Sect. 4.1, there are neutral term with a "bad" shape, such as  $\pi_1 \lambda x. t$  (this term is even hereditary neutral if  $t \in \mathcal{SN}_{\beta\pi}$ ). Therefore, in contrast with the pure  $\lambda$ -calculus, there are neutral terms which are not of the form  $E[t]$  with  $t$  either a variable or a redex. Fortunately, these terms are harmless since they are hereditary neutral when strongly normalizing. A similar property is used in [Tat07], although stated differently. In order to show this, we explicitly work with weak-head  $\beta\pi$ -reduction (see Rem. 3.2.3). Weak-head redexes are called key redexes in [Luo90].

**Definition 6.3.9 (Weak-Head  $\beta\pi$ -Reduction)** The relation  $\rightarrow_{wh}$  of weak-head  $\beta\pi$ -reduction is defined as

$$E[t] \rightarrow_{wh} E[u] \quad \text{if and only if} \quad t \mapsto_{\beta\pi} u.$$

We denote by  $\mathcal{HNF}_{\beta\pi}$  the set of terms which are in weak-head  $\beta\pi$ -normal form.

Note that a weak-head  $\beta\pi$ -reducible term is necessarily neutral. Moreover, neutral terms with a "bad" shape such that  $\pi_1 \lambda x. t$  are in weak-head  $\beta\pi$ -normal form.

As with the pure  $\lambda$ -calculus, the strong principal reduct property is split in two cases, distinguishing hereditary neutral terms from the neutral terms which have at least one value. The first step toward the strong principal reduct property is to show that a neutral term which has at least one value has a weak-head  $\beta\pi$ -redex. In other words, strongly normalizing neutral terms in weak-head  $\beta\pi$ -normal form are hereditary neutral. This is a direct consequence of the fact that the set  $\mathcal{HNF}_{\beta\pi} \cap \mathcal{N}_{\beta\pi}$  is stable by  $\beta\pi$ -reduction. In order to show this property, it is convenient to work with *atomic elimination contexts*  $\epsilon[\ ]$  and *atomic introduction contexts*  $\iota[\ ]$  defined as follows:

$$\epsilon[\ ] ::= [\ ] t \mid \pi_i [\ ] \quad \iota[\ ] ::= \lambda x. [\ ] \mid \langle [\ ], t \rangle \mid \langle t, [\ ] \rangle.$$

It is clear that all values are of the form  $\iota[t]$  and that all  $\beta\pi$ -redexes are of the form  $\epsilon[\iota[t]]$ . We write  $\epsilon[\ ] \perp \iota[\ ]$  when  $\epsilon[\iota[x]]$  is a  $\beta\pi$ -redex for all  $x \in \mathcal{X}$ .

**Proposition 6.3.10** *If  $t \in \mathcal{HNF}_{\beta\pi} \cap \mathcal{N}_{\beta\pi}$  and  $t \rightarrow_{\beta\pi} t'$  then  $t' \in \mathcal{HNF}_{\beta\pi} \cap \mathcal{N}_{\beta\pi}$ .*

PROOF. By induction on  $t$ . The case  $t \in \mathcal{X}$  is trivial. Assume  $t = \epsilon[t_1]$  and let  $t \rightarrow_{\beta\pi} t'$ . Since  $t \in \mathcal{HNF}_{\beta\pi}$ , we have  $t' = \epsilon'[t'_1]$  with  $(t_1, \epsilon[\ ] \rightarrow_{\beta\pi} (t'_1, \epsilon'[\ ])$ , hence  $t' \in \mathcal{N}_{\beta\pi}$ . If  $t_1 \in \mathcal{N}_{\beta\pi}$ , then  $t_1 \in \mathcal{N}_{\beta\pi} \cap \mathcal{HNF}_{\beta\pi}$  and by induction hypothesis  $t'_1 \in \mathcal{N}_{\beta\pi} \cap \mathcal{HNF}_{\beta\pi}$ . It follows that  $t' = \epsilon'[t'_1] \in \mathcal{HNF}_{\beta\pi}$ . Otherwise,  $t_1 = \iota[t_2]$  with  $\epsilon[\ ] \not\rightarrow_{\beta\pi} \iota[\ ]$ . Hence,  $t'_1 = \iota'[t'_2]$  with  $(t_2, \iota[\ ] \rightarrow_{\beta\pi} (t'_2, \iota'[\ ])$ , and  $\epsilon'[\ ] \not\rightarrow_{\beta\pi} \iota'[\ ]$ . Hence  $t' = \epsilon'[\iota'[t'_2]] \in \mathcal{HNF}_{\beta\pi}$ .  $\square$

We now show that weak-head  $\beta\pi$ -reduction gives strong principal reducts. This subsumes Prop. 6.3.5.

**Proposition 6.3.11** *If  $t \rightarrow_{wh} u$  then  $t \sqsubseteq_{\mathcal{V}} u$ .*

PROOF. We show that for all  $n \in \mathbb{N}$ , for all  $E[\ ] \in \mathcal{E}_{\Rightarrow}$ , all  $t, u \in \Lambda(\Sigma)$  such that  $t \mapsto_{\beta\pi} u$  and all  $v \in \mathcal{V}$ ,

$$E[t] \rightarrow_{\beta\pi}^n v \quad \text{implies} \quad E[u] \rightarrow_{\beta\pi}^* v.$$

Note that we must have  $n \geq 1$  since  $E[t]$  is neutral. We reason by induction on  $n$ .

**Base case ( $n = 1$ ).** In this case, we have  $E[t] \rightarrow_{\beta\pi} v$ . If  $v \neq E[u]$ , then by non-interaction (properties (11), (12) and (14)) and weak standardization (properties (15) and (16)), there are  $t', u'$  such that  $v = E'[t']$  with  $(E[\ ], t) \rightarrow_{\beta\pi} (E'[\ ], t')$  and  $t' \mapsto_{\beta\pi} u' \leftarrow_{\beta\pi}^* u$ , hence  $v$  is neutral, which yields a contradiction. It follows that  $v = E[u]$  and we are done.

**Induction case.** Assume that  $E[t] \rightarrow_{\beta\pi} w \rightarrow_{\beta\pi}^n v$ . If  $w \neq E[u]$ , by non-interaction and weak standardization, there are  $t', u'$  such that  $w = E'[t']$  with  $(E[\ ], t) \rightarrow_{\beta\pi} (E'[\ ], t')$  and  $t' \mapsto_{\beta\pi} u' \leftarrow_{\beta\pi}^* u$ . In this case, by induction hypothesis on  $n$  we get  $E'[u'] \rightarrow_{\beta\pi}^* v$ , hence  $E[u] \rightarrow_{\beta\pi}^* v$ .  $\square$

The strong principal reduct property is shown similarly as for the pure  $\lambda$ -calculus (Lem. 6.3.6). We then deduce the stability by union of  $\mathcal{CR}_{\beta\pi}$  using Thm. 6.3.4.

**Lemma 6.3.12 (Strong Principal Reduct Property)** *In the  $\lambda$ -calculus with products, every reducible strongly normalizable neutral term  $t$  has a strong principal reduct.*

PROOF. Let  $t \in \mathcal{N}_{\beta\pi} \cap \mathcal{SN}_{\beta\pi}$  be reducible. If  $t$  is hereditary neutral, then  $\mathcal{V}(t) = \emptyset$ , hence every  $u \in (t)_{\beta\pi}$  is a strong principal reduct of  $t$ . Otherwise, by Prop. 6.3.10 we know that  $t$  has weak-head  $\beta\pi$ -reduct  $u$  and we have  $t \lesssim_{\mathcal{V}} u$  by Prop. 6.3.11.  $\square$

**Corollary 6.3.13**  *$\mathcal{CR}_{\beta\pi}$  is stable by union.*

**Orthogonal constructor rewriting.** We finish this section by discussing the case of constructor rewriting. Recall that given a rewrite system  $\mathcal{R}$  with constructors in  $\mathcal{C}$ , the reducibility candidates  $\mathcal{CR}_{\beta\mathcal{R}\mathcal{C}}$  have been defined in Ex. 4.2.3.

We have seen that weak standardization allows to prove the existence of strong principal reducts for the pure  $\lambda$ -calculus and for the  $\lambda$ -calculus with products. On the other hand, as seen in Ex. 6.3.2 and Ex. 3.5.1, there are confluent rewrite systems which lack weak standardization. This prevents both reducibility candidates from being stable by union and saturated sets from being uniformly defined.

We are looking for a property similar to weak standardization which implies the existence of strong principal reducts. Weak standardization is a particular case of *standardization*, which has been shown for orthogonal first-order rewriting in [HL91], has been generalized to first-order

rewriting in [Bou85] and has been extensively explored and generalized to abstract rewriting in [Mel05]. In our framework, it implies that starting from a neutral term, there are possibly different classes of head redexes, each of them leading to possibly disjoint sets of values.

However, this is too weak to get the strong principal reduct property, because it asks for a canonical way of producing values: even if there can be different principal reducts of a given term, they must all have exactly the same values. Hence, we need that all values of a neutral term can be produced by contracting the *same* head redex. As seen above, this is the case of the  $\lambda$ -calculus with products: if a neutral term has at least one value, then it has a unique (weak) head redex, and any value can be obtained in a derivation starting with the contraction of this redex. Therefore, it is natural to seek for a class of rewrite systems on which standardization implies the existence of strong principal reducts. This is the case of *orthogonal* constructor rewriting. Recall from Def. 2.1.3 that a rewrite system is orthogonal when it is left-linear and has no critical pair. For instance, this is the case of the rewrite system of Ex. 2.1.4.

In contrast with the  $\lambda$ -calculus with products, standardization alone is not exactly what we need to get strong principal reducts. This comes from the fact that orthogonality does not imply sequentiality [HL91]: even if any derivation from a neutral term  $t$  to a value can be reorganized in a derivation starting by a head redex, this reorganization may not be directly readable from the structure of that term. Thus, we need a convenient characterization of head redexes. To this end, we use the notion of *external* redex, developed in the framework of Conditional Combinatory Expression Reduction Systems (CCERS) [KOvO01, GKK05]. Intuitively, a redex is external in a term if its descendants occur under no redex argument in any derivation starting from this term. We can show that contracting an external redex gives a strong principal reduct. Moreover, it has been shown in [KOvO01] that orthogonal CCERS have external redexes. We then obtain the strong principal reduct property for the  $\lambda$ -calculus combined with orthogonal constructor rewriting since it is an orthogonal CCERS.

The proof that the  $\lambda$ -calculus combined with orthogonal constructor rewriting has strong principal reducts leads us outside of the technical scope of this paper. It has been published in [Rib08].

**Theorem 6.3.14 ([Rib08])** *If  $\mathcal{R}$  is an orthogonal rewrite system with constructors in  $\mathcal{C}$ , then  $\mathcal{CR}_{\beta\mathcal{RC}}$  is stable by union.*

## 6.4 Closure by Union of Biorthogonals

We now look at the application of Prop. 6.1.3 to biorthogonals. We begin by considering the general case of a closure operator defined by orthogonality, in the sense of Def. 5.1.4. We then apply these ideas to biorthogonals for reducibility, in the sense of Def. 5.2.1. The content of this section was briefly sketched in [Rib07c].

**The general case.** Consider two sets  $\mathcal{A}$  and  $\Pi$  and a binary relation  $\perp\!\!\!\perp \subseteq \mathcal{A} \times \Pi$ . Recall that by Prop. 5.1.3,  $(\_)^\perp\!\!\!\perp$  is a closure operator. By Lem. 2.2.6, this implies that biorthogonals are stable by intersections. In particular,

$$(A \cap B)^\perp\!\!\!\perp = A^\perp\!\!\!\perp \cap B^\perp\!\!\!\perp = A \cap B \quad \text{for all } A, B \in \mathcal{P}(\mathcal{A})^\perp\!\!\!\perp.$$

However, biorthogonality does not behave that well with union.

**Proposition 6.4.1** *For all  $A, B \in \mathcal{P}(\mathcal{A})$  (resp.  $\mathcal{P}(\Pi)$ ), we have*

$$A^\perp\!\!\!\perp \cap B^\perp\!\!\!\perp = (A \cup B)^\perp\!\!\!\perp, \tag{30}$$

$$A^\perp\!\!\!\perp \cup B^\perp\!\!\!\perp \subseteq (A \cap B)^\perp\!\!\!\perp. \tag{31}$$



PROOF. We begin by (30). First, since  $A, B \subseteq A \cup B$ , we have  $(A \cup B)^\perp \subseteq A^\perp, B^\perp$  thanks to property (27), hence  $(A \cup B)^\perp \subseteq A^\perp \cap B^\perp$ . Conversely, since  $A^\perp \cap B^\perp \subseteq A^\perp, B^\perp$ , by two applications of Prop. 5.1.2 we obtain  $A, B \subseteq (A^\perp \cap B^\perp)^\perp$ , hence  $A \cup B \subseteq (A^\perp \cap B^\perp)^\perp$  and  $(A^\perp \cap B^\perp) \subseteq (A \cup B)^\perp$  by Prop. 5.1.2 again.

Concerning (31), since  $A \cap B \subseteq A, B$ , we have  $A^\perp, B^\perp \subseteq (A \cap B)^\perp$  by property (27), hence  $A^\perp \cup B^\perp \subseteq (A \cap B)^\perp$ .  $\square$

Note that the converse of (31) is not true in general: if  $\pi$  is orthogonal to every element of  $A \cap B$ , in general there is no reason for  $\pi$  to be orthogonal to every element of  $A$  or to every element of  $B$ . It follows from Prop. 6.4.1 that

$$A^{\perp\perp} \cup B^{\perp\perp} \subseteq (A^\perp \cap B^\perp)^\perp = (A \cup B)^{\perp\perp}. \quad (32)$$

But in general  $A^{\perp\perp} \cup B^{\perp\perp} \neq (A \cup B)^{\perp\perp}$ . In words, biorthogonals are in general not stable by union.

**Remark 6.4.2** *The biorthogonal closure of union is quite informative: if  $a \in (A \cup B)^{\perp\perp}$  then  $a \perp \pi$  for all  $\pi \in A^\perp \cap B^\perp$ . Hence, though not stable by union, biorthogonals can nevertheless be interesting in presence of union types [Vou04, VM04, Rib07c].*

Moreover, given biorthogonals  $A$  and  $B$ , there is a nice symmetry between

$$(A \cup B)^{\perp\perp} = (A^\perp \cap B^\perp)^\perp \quad \text{and} \quad (A \cap B)^{\perp\perp} = (A^\perp \cup B^\perp)^\perp$$

(the latter is obtained by applying (30) to  $(A \cap B)^{\perp\perp} = A^{\perp\perp} \cap B^{\perp\perp} = A \cap B$ ).

Similarly as with reducibility candidates, the closure operator  $(\_)^{\perp\perp}$  gives a specialization preorder  $\lesssim_{\perp\perp}$  such that according to Prop. 6.1.3, the closure by union of  $\mathcal{P}^*(\mathcal{A})^{\perp\perp}$  is the set of all non-empty  $A \subseteq \mathcal{A}$  which are downward-closed w.r.t.  $\lesssim_{\perp\perp}$ . Note that we have

$$a \lesssim_{\perp\perp} b \quad \text{if and only if} \quad a^{\perp\perp} \subseteq b^{\perp\perp}.$$

However, we prefer to reason with an other preorder, which gives better intuitions when elements of  $\mathcal{A}$  are seen as "programs" and the elements of  $\Pi$  are seen as "contexts" (i.e. "co-programs").

**Definition 6.4.3** *We define the relation  $\lesssim_\perp \subseteq \mathcal{A} \times \mathcal{A}$  by*

$$a \lesssim_\perp b \quad \text{if and only if} \quad a^\perp \subseteq b^\perp.$$

We thus have  $a \lesssim_\perp b$  if and only if

$$\forall \pi \in \Pi. \quad a \perp \pi \implies b \perp \pi.$$

Seeing  $a \perp \pi$  as a test of program  $a$  against context  $\pi$ , we have  $a \lesssim_\perp b$  if and only if  $b$  succeeds on all tests on which  $a$  succeeds.

The preorders  $\lesssim_\perp$  and  $\lesssim_{\perp\perp}$  are exactly symmetric:

$$a \lesssim_\perp b \quad \text{if and only if} \quad b \lesssim_{\perp\perp} a. \quad (33)$$

This is a direct consequence of the two following properties of  $(\_)^\perp$ , numbered respectively (27) and (28) in Sect. 5, page 37:

$$(X \subseteq Y \implies Y^\perp \subseteq X^\perp) \quad \text{and} \quad X^\perp = X^{\perp\perp\perp}.$$

It follows from Prop. 6.1.3 and (33) that the closure by union of  $\mathcal{P}^*(\mathcal{A})^{\perp\perp}$  is the set of all non-empty  $A \subseteq \mathcal{A}$  which are *upward-closed* w.r.t.  $\lesssim_\perp$ .

**Definition 6.4.4** *Let  $\mathcal{O}_\perp$  be the set of non-empty  $A \subseteq \mathcal{A}$  which are upward-closed w.r.t.  $\lesssim_\perp$ .*



**Biorthogonality for reducibility.** We now look at how these notions apply to reducibility. Let  $\rightarrow_R$  be a rewrite relation and  $\mathcal{E}$  be a set of elimination contexts for  $\rightarrow_R$ . Recall that in Def. 5.2.1 we have defined  $\perp\!\!\!\perp \subseteq \Lambda(\Sigma) \times \mathcal{E}$  as

$$t \perp\!\!\!\perp E[\ ] \quad \text{if and only if} \quad E[t] \in \mathcal{SN}_R.$$

We write  $\lesssim_{\mathcal{SN}}$  for  $\lesssim_{\perp\!\!\!\perp}$  and  $\mathcal{O}_{\mathcal{SN}}$  for  $\mathcal{O}_{\perp\!\!\!\perp}$ .

Recall that by Lem. 5.2.2 we have  $\mathcal{P}^*(\mathcal{SN}_R)^{\perp\!\!\!\perp} \subseteq \mathcal{CR}_{\mathcal{RE}}$ . Moreover, note that  $t \rightarrow_R u$  implies  $t \lesssim_{\mathcal{SN}} u$ . The question, now, is how to ensure  $\mathcal{O}_{\mathcal{SN}} \subseteq \mathcal{CR}_{\mathcal{RE}}$ . The reasoning is analogous to that of Sect. 6.3 with reducibility candidates. Given  $D \in \mathcal{O}_{\mathcal{SN}}$ , it is clear that  $D$  is stable by reduction since

$$\forall t, u. \quad t \rightarrow_R u \implies t \lesssim_{\mathcal{SN}} u.$$

The case of  $(\mathcal{CR}1)$  is less clear. Consider a strongly-normalizing reducible neutral term  $t$  and assume that  $D$  is the set  $D(t) \in \mathcal{O}_{\mathcal{SN}}$  defined as

$$D(t) \stackrel{\text{def}}{=} \{v \mid \exists u \in (t)_R. \quad u \lesssim_{\mathcal{SN}} v\}.$$

In order to have  $t \in D(t)$ , there must be a term  $u \in (t)_R$  such that  $u \lesssim_{\mathcal{SN}} t$ . We say that  $u$  is a *principal reduct* of  $t$ .

**Definition 6.4.5 (Principal Reduct)** *Given a reducible term  $t \in \mathcal{N}_{\mathcal{RE}} \cap \mathcal{SN}_R$ , a term  $u \in (t)_R$  such that  $u \lesssim_{\mathcal{SN}} t$  is a principal reduct of  $t$ .*

*We say that  $\rightarrow_R$  satisfies the principal reduct property or that  $\rightarrow_R$  has principal reducts when every reducible strongly normalizing neutral term has a principal reduct.*

As for reducibility candidates, with which the existence of strong principal reducts characterizes that  $\mathcal{O}_{\mathcal{V}} \subseteq \mathcal{CR}_{\mathcal{RE}}$ , the existence of principal reducts characterizes that  $\mathcal{O}_{\mathcal{SN}} \subseteq \mathcal{CR}_{\mathcal{RE}}$ .

**Theorem 6.4.6** *The following are equivalent:*

- (i)  $\mathcal{O}_{\mathcal{SN}} \subseteq \mathcal{CR}_{\mathcal{RE}}$ ,
- (ii) every reducible term  $t \in \mathcal{N}_{\mathcal{RE}} \cap \mathcal{SN}_R$  has a principal reduct.

PROOF.

**(i)  $\implies$  (ii).** Let  $t \in \mathcal{N}_{\mathcal{RE}} \cap \mathcal{SN}_R$  be reducible. By assumption, the set  $D(t)$  defined above is a reducibility candidate, hence  $t \in D(t)$ . It follows that there is  $u \in (t)_R$  such that  $u \lesssim_{\mathcal{SN}} t$ .

**(ii)  $\implies$  (i).** Consider  $C \in \mathcal{O}_{\mathcal{SN}}$  and let us check that it is a reducibility candidate.

- $(\mathcal{CR}0)$ . If  $t \in C$  and  $t \rightarrow_R u$  then  $u \in \mathcal{SN}_R$  and  $t \lesssim_{\mathcal{SN}} u$  hence  $u \in C$ .
- $(\mathcal{CR}1)$ . Let  $t \in \mathcal{N}_{\mathcal{RE}}$  such that  $(t)_R \subseteq C$ . Note that  $t \in \mathcal{SN}_R$  since  $C \subseteq \mathcal{SN}_R$ .  
If  $(t)_R = \emptyset$  then  $t \in \mathcal{HN}_{\mathcal{RE}}$ , hence  $t \in C$  by Prop. 6.4.12. Otherwise, by assumption there is some  $u \in (t)_R$  such that  $u \lesssim_{\mathcal{SN}} t$ , hence  $t \in C$  because  $(t)_R \subseteq C$ .  $\square$

Note that there exist rewrite relations  $\rightarrow_R$  such that there are reducible  $t \in \mathcal{SN}_R \cap \mathcal{N}_{\mathcal{RE}}$  which have no principal reduct. In fact, this was exactly the problem underlined in Ex. 3.5.1 concerning the definition of saturated sets for rewriting. We recall it here.

**Example 6.4.7** Consider the confluent system

$$p \mapsto_{\mathcal{R}} \lambda x. \lambda y. \lambda z. g(x y) \quad p \mapsto_{\mathcal{R}} \lambda x. \lambda y. \lambda z. g(x z) \quad g(x) \mapsto_{\mathcal{R}} a$$

We have  $p \in \mathcal{SN}_{\beta\mathcal{R}}$ , but there are untyped elimination contexts which separate the terms

$$\lambda x. \lambda y. \lambda z. g(x y) \quad \text{and} \quad \lambda x. \lambda y. \lambda z. g(x z)$$

with respect to strong normalization. For instance, we have

$$\begin{aligned} (\lambda x. \lambda y. \lambda z. g(x y)) \delta a \delta &\in \mathcal{SN}_{\beta\mathcal{R}} & (\lambda x. \lambda y. \lambda z. g(x z)) \delta a \delta &\notin \mathcal{SN}_{\beta\mathcal{R}} \\ (\lambda x. \lambda y. \lambda z. g(x y)) \delta \delta a &\notin \mathcal{SN}_{\beta\mathcal{R}} & (\lambda x. \lambda y. \lambda z. g(x z)) \delta \delta a &\in \mathcal{SN}_{\beta\mathcal{R}} \end{aligned}$$

where  $\delta =_{\text{def}} \lambda x. x x$ , hence  $\delta \delta \notin \mathcal{SN}_{\beta}$ . This means that the terms

$$\lambda x. \lambda y. \lambda z. g(x y) \quad \text{and} \quad \lambda x. \lambda y. \lambda z. g(x z)$$

are not comparable w.r.t.  $\lesssim_{\mathcal{SN}}$ , hence that  $p$  has no principal reduct.

The principal reduct property is very close to the strong principal reduct property defined in Def. 6.3.3. The names come from the fact that every strong principal reduct is a principal reduct. This follows from the next proposition.

**Proposition 6.4.8** For all  $t, u \in \mathcal{SN}_{\mathcal{R}}$ , we have

$$t \lesssim_{\mathcal{V}} u \implies u \lesssim_{\mathcal{SN}} t.$$

PROOF. Let  $u \in \mathcal{SN}_{\mathcal{R}}$ . We show that for all  $t \lesssim_{\mathcal{V}} u$  and all  $E[\ ] \in u^{\perp}$ , we have  $E[t] \in \mathcal{SN}_{\mathcal{R}}$ . We reason by induction on pairs  $(E[\ ], t)$  ordered by the product extension of  $\rightarrow_{\mathcal{R}}$ .

Let  $v$  such that  $E[t] \rightarrow_{\mathcal{R}} v$ . There are two cases.

- If  $t \in \mathcal{V}$ , then  $u \rightarrow_{\mathcal{R}}^* t$  since  $t \lesssim_{\mathcal{V}} u$ . Hence  $E[u] \rightarrow_{\mathcal{R}}^* v$  and  $v \in \mathcal{SN}_{\mathcal{R}}$ .
- Otherwise, we have  $v = E'[t']$  with  $(E[\ ], t) \rightarrow_{\mathcal{R}} (E'[\ ], t')$ . If  $E[\ ] \rightarrow_{\mathcal{R}} E'[\ ]$ , we have  $v = E'[t] \in \mathcal{SN}_{\mathcal{R}}$  by induction hypothesis, since  $E[u] \rightarrow_{\mathcal{R}} E'[u] \in \mathcal{SN}_{\mathcal{R}}$ . Otherwise, we have  $v = E[t']$  with  $t \rightarrow_{\mathcal{R}} t'$ . Since  $t' \lesssim_{\mathcal{V}} t$ , we get  $t' \lesssim_{\mathcal{V}} u$ , hence  $v = E[t'] \in \mathcal{SN}_{\mathcal{R}}$  by induction hypothesis.  $\square$

The converse of the above property is not true in general, even for the pure  $\lambda$ -calculus. This implies that in general, we do not have  $\mathcal{CR}_{\mathcal{R}\mathcal{E}} \subseteq \mathcal{O}_{\mathcal{SN}}$ .

**Example 6.4.9**

- We have  $[\lambda x. y] \vec{t} \in \mathcal{SN}_{\beta}$  for all  $[\ ] \vec{t} \in \mathcal{SN}_{\beta}$ . Hence  $\lambda x. x \lesssim_{\mathcal{SN}} \lambda x. y$ , whereas  $\lambda x. x$  and  $\lambda x. y$  are not comparable w.r.t.  $\lesssim_{\mathcal{V}}$ .  
It follows that  $\lambda x. y \notin \mathcal{CR}(\lambda x. x)$ , hence that  $\mathcal{CR}(\lambda x. x)$  is not upward-closed w.r.t.  $\lesssim_{\mathcal{SN}}$ . This implies that  $\mathcal{CR}_{\beta} \not\subseteq \mathcal{O}_{\mathcal{SN}}$ .
- There are rewrite system which have principal reducts but not strong principal reducts. This is the case of the rewrite system of Ex. 6.3.2:

$$p \mapsto_{\mathcal{R}} \lambda x. c_1 \quad p \mapsto_{\mathcal{R}} \lambda x. c_2 \quad c_i \mapsto_{\mathcal{R}} d.$$

The terms  $\lambda x. c_1$  and  $\lambda x. c_2$  are not comparable w.r.t.  $\lesssim_{\mathcal{V}}$  but they are equivalent w.r.t.  $\simeq_{\mathcal{SN}}$ . Hence  $p$  has principal reducts and  $\mathcal{O}_{\mathcal{SN}} \subseteq \mathcal{CR}_{\beta\pi\mathcal{R}}$ .

Since Prop. 6.4.8 implies that every strong principal reduct is a principal reduct, by Thm. 6.3.4 and Thm. 6.4.6, we get that  $\mathcal{O}_{\mathcal{SN}} \subseteq \mathcal{CR}_{\mathcal{RE}}$  whenever  $\mathcal{CR}_{\mathcal{RE}}$  is stable by union.

**Corollary 6.4.10** *If  $\mathcal{CR}_{\mathcal{RE}}$  is stable by union then  $\mathcal{O}_{\mathcal{SN}} \subseteq \mathcal{CR}_{\mathcal{RE}}$ .*

**Remark 6.4.11** *One can wonder whether the equivalence relations*

$$\simeq_{\mathcal{N}} =_{\text{def}} \lesssim_{\mathcal{N}} \cap \gtrsim_{\mathcal{N}} \quad \text{and} \quad \simeq_{\mathcal{SN}} =_{\text{def}} \lesssim_{\mathcal{SN}} \cap \gtrsim_{\mathcal{SN}}$$

*issued from the preorders  $\lesssim_{\mathcal{V}}$  and  $\lesssim_{\mathcal{SN}}$  could be extended to coherent congruency on  $\Lambda(\Sigma)$ .*

*This is not possible for pure  $\lambda$ -calculus. Indeed,  $\simeq_{\mathcal{V}}$  identifies all hereditary neutral terms, including those which are not  $\beta\eta$ -convertible. (by Prop. 6.4.8,  $\simeq_{\mathcal{SN}}$  identifies more than  $\simeq_{\mathcal{V}}$ ). Therefore, by Böhm's theorem [Böh68] (see also [Kri90]), these congruences identify all pure  $\lambda$ -terms.*

**The reducibility family  $\mathcal{O}_{\mathcal{SN}}$ .** We have given in Thm. 6.4.6 a necessary and sufficient condition to get  $\mathcal{O}_{\mathcal{SN}} \subseteq \mathcal{CR}$ , and we have shown that this condition is met when reducibility candidates are stable by union. On the other hand, we have shown in Sect. 6.3 that it is actually the case for the combination of  $\lambda$ -calculus with orthogonal constructor rewriting. Hence, there are cases where  $\mathcal{O}_{\mathcal{SN}} \subseteq \mathcal{CR}$ . It is therefore tempting to use  $\mathcal{O}_{\mathcal{V}}$  as a reducibility family. In this paragraph, we show that this is possible, under some conditions on elimination contexts.

First, it is clear that  $\mathcal{SN}_{\mathcal{R}} \in \mathcal{O}_{\mathcal{SN}}$ . Moreover, we have  $\mathcal{HN}_{\mathcal{RE}} \subseteq \mathcal{C}$  for all  $\mathcal{C} \in \mathcal{O}_{\mathcal{SN}}$ .

**Proposition 6.4.12** *If  $\mathcal{C} \in \mathcal{O}_{\mathcal{SN}}$  then  $\mathcal{HN}_{\mathcal{RE}} \subseteq \mathcal{C}$ .*

PROOF. If  $t \in \mathcal{HN}_{\mathcal{RE}}$  then by Prop. 4.4.4 we have  $E[t] \in \mathcal{HN}_{\mathcal{RE}} \subseteq \mathcal{SN}_{\mathcal{R}}$  for all  $E[\ ] \in \mathcal{E} \cap \mathcal{SN}_{\mathcal{R}}$ . It follows that  $u \lesssim_{\mathcal{SN}} t$  for all  $u \in \mathcal{SN}_{\mathcal{R}}$ , hence that  $t \in \mathcal{C}$  since  $\mathcal{C}$  is not empty.  $\square$

Furthermore, if  $\mathcal{E}$  contains and is stable by pre-composition with the elimination contexts  $\mathcal{E}_{\Rightarrow \times}$ , then the function space and the binary product constructors preserve  $\mathcal{O}_{\mathcal{SN}}$ .

This relies on the following simple property of  $\lesssim_{\mathcal{SN}}$ . Let  $E[\ ] \in \mathcal{E}$  such that for all  $F[\ ] \in \mathcal{E}$ , we have  $F[E[\ ]] \in \mathcal{E}$ . Then,

$$t \lesssim_{\mathcal{SN}} u \quad \text{implies} \quad E[t] \lesssim_{\mathcal{SN}} E[u]. \quad (34)$$

**Lemma 6.4.13** *Let  $\rightarrow_{\mathcal{R}}$  be a rewrite relation on  $\Lambda(\Sigma)$ .*

- (i) *Let  $\mathcal{E}$  be a set of elimination contexts for  $\rightarrow_{\mathcal{R}}$  such that for all  $t \in \Lambda(\Sigma)$ , we have  $[\ ]t \in \mathcal{E}$  and  $F[[\ ]t] \in \mathcal{E}$  for all  $F[\ ] \in \mathcal{E}$ . Then  $A \Rightarrow B \in \mathcal{O}_{\mathcal{SN}}$  for all  $A, B \in \mathcal{O}_{\mathcal{SN}}$ .*
- (ii) *Let  $\mathcal{E}$  be a set of elimination contexts for  $\rightarrow_{\mathcal{R}}$  such that for all  $i \in \{1, 2\}$ , we have  $\pi_i[\ ] \in \mathcal{E}$  and  $F[\pi_i[\ ]] \in \mathcal{E}$  for all  $F[\ ] \in \mathcal{E}$ . Then  $A \times B \in \mathcal{O}_{\mathcal{SN}}$  for all  $A, B \in \mathcal{O}_{\mathcal{SN}}$ .*

PROOF. We only detail (i) because (ii) is similar and simpler.

To get  $A \Rightarrow B \subseteq \mathcal{SN}_{\mathcal{R}}$ , we reason as in Prop. 3.2.10, using that  $\mathcal{HN}_{\mathcal{RE}} \subseteq \mathcal{A}$  by Prop 6.4.12.

Moreover,  $A \Rightarrow B$  is upward-closed w.r.t.  $\lesssim_{\mathcal{SN}}$ : since the assumptions allow to apply (34), we get that  $t \lesssim_{\mathcal{SN}} u$  implies  $tv \lesssim_{\mathcal{SN}} uv$  for all  $v \in \Lambda(\Sigma)$ .

Finally,  $A \Rightarrow B$  is not empty: given  $t \in \mathcal{HN}_{\mathcal{RE}}$ , for all  $u \in A \subseteq \mathcal{SN}_{\mathcal{R}}$  we have  $tu \in \mathcal{HN}_{\mathcal{RE}}$  by Prop. 4.4.4, hence  $tu \in B$  by Prop. 6.4.12.  $\square$

The hypothesis of Lem. 6.4.13 are satisfied by the elimination contexts  $\mathcal{E}_{\Rightarrow \times}$ . Moreover, the sets  $C \in \mathcal{O}_{\mathcal{SN}}$  satisfy clauses  $(\mathcal{SAT}1)$ ,  $(\mathcal{SAT}2_\beta)$  and  $(\mathcal{SAT}2_{\pi_i})$ . Indeed, strongly normalizing terms of the form  $E[x]$  are hereditary neutral, and we have

$$t_1[t_2/x] \lesssim_{\mathcal{SN}} (\lambda x. t_1)t_2 \quad \text{and} \quad t_i \lesssim_{\mathcal{SN}} \pi_i \langle t_1, t_2 \rangle \quad \text{for all} \quad t_1, t_2 \in \mathcal{SN}_{\beta\pi}$$

by reasoning as in Lem. 3.3.2, using weak standardization and non-interaction. Hence, reasoning as in the end of Sect. 4.4, we get that  $\mathcal{O}_{\mathcal{SN}}$  is a reducibility family which leads to an adequate type interpretation for the  $\lambda$ -calculus with products.

## 7 Back to Reducibility Candidates and Saturated Sets

This section is devoted to the application of the results on stability by union to the exploration of reducibility candidates and saturated sets. We begin by giving a more precise result on how the preorder  $\lesssim_{\mathcal{V}}$  characterizes the membership of a term to a reducibility candidate. We then compare reducibility candidates and the saturated sets which are stable by reduction. This comparison was first presented in [Rib07b] and strengthen known results [Gal89, Luo90].

This leads us to general notion of saturated sets  $\mathcal{SAT}_{\mathcal{RE}}$  which only applies when  $\rightarrow_{\mathcal{R}}$  has strong principal reduces. We get a general correspondence between  $\mathcal{CR}_{\mathcal{RE}}$  and  $\mathcal{SAT}_{\mathcal{RE}}$ , stating that  $\mathcal{CR}_{\mathcal{RE}}$  is exactly the set  $\mathcal{SAT}_{\mathcal{RE}}^*$  of  $S \in \mathcal{SAT}_{\mathcal{RE}}$  which are stable by reduction. However, even in the case of the pure  $\lambda$ -calculus, the instance  $\mathcal{SAT}_{\beta\mathcal{E}\Rightarrow}$  of  $\mathcal{SAT}_{\mathcal{RE}}$  does not correspond to the usual saturated sets  $\mathcal{SAT}_\beta$ . A precise correspondence may need a notion such that of external redexes [KOv001], but this seems too syntactical to be included in our framework yet.

### 7.1 Reducibility Candidates

In this section, we strengthen Thm. 4.5.14 using the material developed to compare the stability by union of reducibility candidates and the reducibility family  $\mathcal{O}_{\mathcal{SN}}$ . Recall that Thm. 4.5.14 states that reducibility candidates are downward-closed w.r.t.  $\lesssim_{\mathcal{V}}$ . This means that for all  $C \in \mathcal{CR}_{\mathcal{RE}}$  and all  $t, u \in \Lambda(\Sigma)$ ,

$$(t \lesssim_{\mathcal{V}} u \quad \text{and} \quad u \in C) \quad \text{implies} \quad t \in C .$$

In fact, the observational preorder  $\lesssim_{\mathcal{V}}$  captures information on reducibility in a quite stronger sense. First, Prop. 6.4.8 can be rephrased as

$$(t \lesssim_{\mathcal{V}} u \quad \text{and} \quad E[u] \in \mathcal{SN}_{\mathcal{R}}) \quad \text{implies} \quad E[t] \in \mathcal{SN}_{\mathcal{R}} . \quad (35)$$

On the other hand, Prop. 4.5.11 says that

$$t \sqsubseteq_{\mathcal{V}} u \quad \text{implies} \quad E[t] \sqsubseteq_{\mathcal{V}} E[u] \quad (36)$$

(recall that  $\lesssim_{\mathcal{V}}$  is the restriction of  $\sqsubseteq_{\mathcal{V}}$  to  $\mathcal{SN}_{\mathcal{R}} \times \mathcal{SN}_{\mathcal{R}}$ ). Combining (35) with (36), we get

$$(t \lesssim_{\mathcal{V}} u \quad \text{and} \quad E[u] \in \mathcal{SN}_{\mathcal{R}}) \quad \text{implies} \quad E[t] \lesssim_{\mathcal{V}} E[u] . \quad (37)$$

Using the downward-closure of  $\mathcal{CR}_{\mathcal{RE}}$  (Thm. 4.5.14), we arrive at the following property, which can be seen as merging of Thm. 4.5.14 and Lem. 4.2.5.

**Corollary 7.1.1** *For all  $C \in \mathcal{CR}_{\mathcal{RE}}$ ,*

$$(t \lesssim_{\mathcal{V}} u \quad \text{and} \quad E[u] \in C) \quad \text{implies} \quad E[t] \in C .$$

## 7.2 A Comparison of Reducibility Candidates and Saturated Sets

In this section, we compare Girard's reducibility candidates and Tait's saturated sets using the material developed in Sect. 6. We show that the strong principal reduct property allows to give a precise correspondence between  $\mathcal{CR}_\beta$  and the saturated sets  $S \in \mathcal{SAT}_\beta$  which are stable by reduction. For the  $\lambda$ -calculus with products, we show that we need saturated sets different from  $\mathcal{SAT}_{\beta\pi}$  to get the same correspondence.

**Pure lambda-calculus.** The stability by union of  $\mathcal{CR}_\beta$  relies on a strong connection between  $\mathcal{CR}_\beta$  and the set  $\mathcal{SAT}_\beta$  of  $\beta$ -saturated sets. Recall that these saturated sets were defined Def. 3.2.6 as the set of all  $S \in \mathcal{SN}_\beta$  such that

- (SAT1) if  $E[\ ] \in \mathcal{SN}_\beta \cap \mathcal{E}_\Rightarrow$  and  $x \in \mathcal{X}$  then  $E[x] \in S$ ,
- (SAT2 $_\beta$ ) if  $E[t[u/x]] \in S$  and  $u \in \mathcal{SN}_\beta$  then  $E[(\lambda x.t)u]$ .

In Lem. 4.2.6, we have shown that every  $C \in \mathcal{CR}_\beta$  satisfies the clauses (SAT1) and (SAT2 $_\beta$ ), hence that  $C \in \mathcal{SAT}_\beta$ . It is interesting to note that the proof for (SAT2 $_\beta$ ) hides the property shown in Prop. 6.3.5. In order to see why, given  $C \in \mathcal{CR}_\beta$ , let us show that  $C$  satisfies (SAT2 $_\beta$ ). Therefore, let  $E[\ ] \in \mathcal{E}_\Rightarrow \cap \mathcal{SN}_\beta$  and  $t, u \in \mathcal{SN}_\beta$ . We have to show that

$$E[t[u/x]] \in C \quad \text{implies} \quad E[(\lambda x.t)u] \in C .$$

According to Thm. 4.5.14, taking

$$C \stackrel{\text{def}}{=} \mathcal{CR}(E[t[u/x]]) ,$$

this amount to showing that  $E[(\lambda x.t)u] \in \mathcal{SN}_\beta$  and

$$E[(\lambda x.t)u] \lesssim_\nu E[t[u/x]] .$$

In words,  $E[t[u/x]]$  is a strong principal reduct of  $E[(\lambda x.t)u]$ .

The stability by union of  $\mathcal{CR}_\beta$  can be rephrased as a precise correspondence between  $\mathcal{CR}_\beta$  and a subset of the saturated sets  $\mathcal{SAT}_\beta$ : reducibility candidates are exactly the saturated sets which are stable by reduction. Recall that stability by reduction is not satisfied by all  $S \in \mathcal{SAT}_\beta$ , as shown in Ex. 3.5.3.

**Definition 7.2.1** Let  $\mathcal{SAT}_\beta^*$  be the set of all  $S \in \mathcal{SAT}_\beta$  such that

- (SAT0) if  $t \in S$  and  $t \rightarrow_\beta u$  then  $u \in S$ .

**Lemma 7.2.2**  $\mathcal{CR}_\beta = \mathcal{SAT}_\beta^*$ .

PROOF. If  $C \in \mathcal{CR}_\beta$  then  $C$  satisfies (SAT0) by (CR0), and the clauses (SAT1), (SAT2 $_\beta$ ) follow from Lem. 4.2.6.

Conversely, if  $S \in \mathcal{SAT}_\beta^*$  then  $S$  satisfies (CR0). Consider the case of (CR1). If  $t$  is a neutral term such that  $(t)_\beta \subseteq S$ , then we have  $t \in \mathcal{SN}_\beta$ . Moreover, either  $t = E[x]$  and  $t \in S$  by (SAT1), or  $t = E[(\lambda x.u)v]$  and we get  $t \in S$  by (SAT2 $_\beta$ ) since  $E[u[v/x]] \in S$ .  $\square$

This gives a second proof that  $\mathcal{CR}_\beta$  is stable by union: since the clause (SAT0) is preserved by union, we get the stability by union of  $\mathcal{SAT}_\beta^*$  by Thm. 6.2.1, hence the stability by union of  $\mathcal{CR}_\beta$  by Lem. 7.2.2.

**Lambda-calculus with products.** We now turn to the case of  $\lambda$ -calculus with products. In contrast with the pure  $\lambda$ -calculus, the closure by reduction of  $\mathcal{SAT}_{\beta\pi}$  is not  $\mathcal{CR}_{\beta\pi}$ . This is due to the bad shape of some neutral terms in  $\mathcal{EN}_{\beta\pi}$ , already noticed in Sect. 4.1 and Sect. 6.3: since  $\mathcal{HN}_{\beta\pi}$  contains terms not of the form  $E[x]$  with  $E[\ ] \in \mathcal{E}_{\Rightarrow \times}$ ,  $\mathcal{HN}_{\beta\pi}$  is not the least element of  $\mathcal{SAT}_{\beta\pi}$ . Therefore, we use a modified notion saturated sets  $\mathcal{SAT}_{\beta\pi}^m$  in which the usual clause ( $\mathcal{SAT}1$ ) is subsumed by requiring that  $\mathcal{HN}_{\beta\pi}$  is the least element of  $\mathcal{SAT}_{\beta\pi}^m$ .

**Definition 7.2.3** *We let  $\mathcal{SAT}_{\beta\pi}^m$  be the set of all  $S \in \mathcal{SAT}_{\beta\pi}$  such that  $\mathcal{HN}_{\beta\pi} \subseteq S$  and*

*( $\mathcal{SAT}0$ ) if  $t \in S$  and  $t \rightarrow_{\beta\pi} u$  then  $u \in S$ .*

Note that  $\mathcal{SAT}_{\beta\pi}^m$  is not empty since it contains  $\mathcal{SN}_{\beta\pi}$ . As for the stability by union of  $\mathcal{CR}_{\beta\pi}$  in Sect. 6.3, "bad" neutral terms are dealt with thanks to Prop. 6.3.10.

**Lemma 7.2.4**  $\mathcal{CR}_{\beta\pi} = \mathcal{SAT}_{\beta\pi}^m$ .

PROOF. If  $C \in \mathcal{CR}_{\beta\pi}$  then  $C$  satisfies the clause ( $\mathcal{SAT}0$ ) by ( $\mathcal{CR}0$ ), and the clauses ( $\mathcal{SAT}2_\beta$ ), ( $\mathcal{SAT}2_{\pi i}$ ) by Lem 4.2.6. Moreover, we have  $\mathcal{HN}_{\beta\pi} \subseteq C$  by Prop. 4.4.2.

Conversely, if  $S \in \mathcal{SAT}_{\beta\pi}^m$  then  $S$  satisfies ( $\mathcal{CR}0$ ). For ( $\mathcal{CR}1$ ), let  $t$  be a neutral term such that  $(t)_{\beta\pi} \subseteq S$ . Note that  $t \in \mathcal{SN}_{\beta\pi}$ . If  $t \in \mathcal{HN}_{\beta\pi}$  then  $t \in S$  by definition. Otherwise, by Prop 6.3.10,  $t$  is of the form  $E[u]$  where  $u$  is a  $\beta\pi$ -redex and we conclude by ( $\mathcal{SAT}2_\beta$ ) and ( $\mathcal{SAT}2_{\pi i}$ ).  $\square$

### 7.3 Toward a General Notion of Saturated Sets

We have seen that proving the stability by union of  $\mathcal{CR}_\beta$  amounts to showing that  $\mathcal{CR}_\beta$  is the set  $\mathcal{SAT}_\beta^*$  of all  $S \in \mathcal{SAT}_\beta$  which are stable by reduction. This suggests a connection between the stability by union reducibility candidates and conditions to get sound saturated sets.

In this section, we propose a general notion of saturated sets  $\mathcal{SAT}_{\mathcal{RE}}$  based on the strong principal reduct property. As with reducibility candidates  $\mathcal{CR}_{\mathcal{RE}}$ , these saturated sets only depend on a rewrite relation  $\rightarrow_{\mathcal{R}}$  and a set of elimination contexts  $\mathcal{E}$ . In contrast with reducibility candidates, they are applicable only when  $\rightarrow_{\mathcal{R}}$  has the strong principal reduct property (that is, strictly less often than the reducibility family  $\mathcal{O}_{\mathcal{SN}}$ ). We show that in the case of the  $\lambda$ -calculus with products, these saturated sets give an adequate type interpretation. Moreover, we show that  $\mathcal{CR}_{\mathcal{RE}}$  is exactly the set  $\mathcal{SAT}_{\mathcal{RE}}^*$  of all  $S \in \mathcal{SAT}_{\mathcal{RE}}$  which are stable by reduction. However, even in the case of the pure  $\lambda$ -calculus, the instance  $\mathcal{SAT}_{\beta\mathcal{E} \Rightarrow}$  of  $\mathcal{SAT}_{\mathcal{RE}}$  does not correspond to the usual saturated sets  $\mathcal{SAT}_\beta$  (see Rem. 7.3.2).

Assume given a rewrite relation  $\rightarrow_{\mathcal{R}}$  and a set of elimination contexts  $\mathcal{E}$  such that  $\rightarrow_{\mathcal{R}}$  has strong principal reducts.

**Definition 7.3.1** *The set  $\mathcal{SAT}_{\mathcal{RE}}$  is the set of all  $S \subseteq \mathcal{SN}_{\mathcal{R}}$  such that*

*( $\mathcal{SAT}1$ )  $\mathcal{HN}_{\mathcal{RE}} \subseteq S$ ,*

*( $\mathcal{SAT}2$ ) for all  $t \in \mathcal{N}_{\mathcal{RE}} \cap \mathcal{SN}_{\mathcal{R}}$ , if  $t \rightarrow_{\mathcal{R}} u$  with  $t \lesssim_{\mathcal{V}} u$  and  $u \in S$  then  $t \in S$ .*

In order to make no confusion with  $\mathcal{SAT}_\beta$  and  $\mathcal{SAT}_{\beta\pi}$ , defined respectively in Def. 3.2.6 and Def. 3.3.1, we write  $\mathcal{SAT}_{\beta\mathcal{E} \Rightarrow}$  and  $\mathcal{SAT}_{\beta\pi\mathcal{E} \Rightarrow \times}$  for the respective instantiations of  $\mathcal{SAT}_{\mathcal{RE}}$  to the pure  $\lambda$ -calculus and the  $\lambda$ -calculus with products.

**Remark 7.3.2** For the pure  $\lambda$ -calculus, we do not have  $\text{SAT}_\beta = \text{SAT}_{\beta\mathcal{E}\Rightarrow}$  : as shown in Ex.6.3.8, a term  $t \in \mathcal{N}_\beta \setminus \mathcal{HN}_\beta$  can have a strong principal  $u$  reduct which is not a weak-head reduct. Hence  $t \in \text{SAT}_{\beta\mathcal{E}\Rightarrow}(u)$  while  $t \notin \text{SAT}_\beta(u)$ .

To get a precise correspondence, it is possible that a syntactical notion such that of external redexes [KOvO01] can be used instead of the strong principal reduct property. However, it is not clear yet how this can fit in our framework, because this notion seems to need a precise syntactical knowledge on rewrite relations, such as a nesting preorder between redexes [KOvO01, Mel05].

We now show that the function space  $- \Rightarrow -$  and the product space  $- \times -$  preserve  $\text{SAT}_{\mathcal{RE}}$  under the same conditions as for  $\mathcal{CR}$  in Lem. 4.4.5. The proof relies on property (37), shown in Sect. 7.1: given  $t, u \in \mathcal{SN}_R$  and  $E[\ ] \in \mathcal{E}$ ,

$$(t \lesssim_{\mathcal{V}} u \quad \text{and} \quad E[u] \in \mathcal{SN}_R) \quad \text{implies} \quad E[t] \lesssim_{\mathcal{V}} E[u] .$$

**Lemma 7.3.3**

(i) If  $[ \ ] t \in \mathcal{E}$  for all  $t \in \Lambda(\Sigma)$ , then

$$A, B \in \text{SAT}_{\mathcal{RE}} \implies A \Rightarrow B \in \text{SAT}_{\mathcal{RE}} .$$

(ii) If  $\pi_i [ \ ] \in \mathcal{E}$  for all  $i \in \{1, 2\}$ , then

$$A, B \in \text{SAT}_{\mathcal{RE}} \implies A \times B \in \text{SAT}_{\mathcal{RE}} .$$

PROOF. We only detail (i) because (ii) is similar and simpler. To get  $A \Rightarrow B \subseteq \mathcal{SN}_R$ , we reason as in Prop. 3.2.10, using that  $\mathcal{HN}_{\mathcal{RE}} \subseteq A$  by (SAT1).

(SAT0) As for (CR0) in Prop. 4.4.5.

(SAT1) Given  $t \in \mathcal{HN}_{\mathcal{RE}}$ , for all  $u \in A \subseteq \mathcal{SN}_R$  we have  $tu \in \mathcal{HN}_{\mathcal{RE}}$  by Prop. 4.4.4, hence  $tu \in B$  by Prop. 6.4.12.

(SAT2) Let  $t \in \mathcal{N}_{\mathcal{RE}} \cap \mathcal{SN}_R$  be reducible and let  $u \in (t)_R$  such that  $t \lesssim_{\mathcal{V}} u$  and  $u \in A \Rightarrow B$ . Given  $v \in A \subseteq \mathcal{SN}_R$ , we have  $uv \in B \subseteq \mathcal{SN}_R$ , hence  $tv \lesssim_{\mathcal{V}} uv$  by (37). We deduce that  $tv \in B$  since  $tv \rightarrow_R uv$ .  $\square$

The hypothesis of Lem. 7.3.3 are satisfied by the elimination contexts  $\mathcal{E}_{\Rightarrow \times}$ . Moreover, since the  $\lambda$ -calculus with products has strong principal reducts (Lem. 6.3.12), every  $S \in \text{SAT}_{\beta\pi\mathcal{E}\Rightarrow \times}$  satisfies the clauses (SAT2 $_\beta$ ) and (SAT2 $_{\pi}$ ). Hence,  $\text{SAT}_{\beta\pi\mathcal{E}\Rightarrow \times}$  is a reducibility family which leads to an adequate type interpretation for the  $\lambda$ -calculus with products.

Our last point is that  $\mathcal{CR}_{\mathcal{RE}}$  is the of all  $S \in \text{SAT}_{\mathcal{RE}}$  which are stable by reduction.

**Definition 7.3.4** Let  $\text{SAT}_{\mathcal{RE}}^*$  be the set of all  $S \in \text{SAT}_{\mathcal{RE}}$  such that

(SAT0) if  $t \in S$  and  $t \rightarrow_R u$  then  $u \in S$ .

**Lemma 7.3.5**  $\mathcal{CR}_{\mathcal{RE}} = \text{SAT}_{\mathcal{RE}}^*$ .

PROOF. Let  $C \in \mathcal{CR}_{\mathcal{RE}}$ . The clause (SAT0) follows from (CR0) and (SAT1) follows from Prop. 4.4.2. For (SAT2), given a reducible term  $t \in \mathcal{N}_{\mathcal{RE}} \cap \mathcal{SN}_R$  and a term  $u \in (t)_R$  such that  $t \lesssim_{\mathcal{V}} u$  and  $u \in C$ , we have  $t \in C$  by Thm. 4.5.14.

Conversely, let  $S \in \text{SAT}_{\mathcal{RE}}^*$ . The clause (CR0) follows from (SAT0). For (CR1), let  $t$  be a neutral term such that  $(t)_R \subseteq S$ . If  $t \in \mathcal{HN}_{\mathcal{RE}}$ , then  $t \in S$  by (SAT1). Otherwise,  $\mathcal{V}(t) \neq \emptyset$ , hence  $t$  is reducible since  $t$  is neutral. By assumption,  $t$  has a strong principal reduct  $u$ . We have  $u \in S$  since  $(t)_R \subseteq S$  and we conclude that  $t \in S$  by (SAT2).  $\square$



Hence, by Lem. 7.2.2, we get that  $\mathcal{SAT}_{\beta\mathcal{E}\Rightarrow}^* = \mathcal{SAT}_{\beta}^*$  and by Lem. 7.2.4 that  $\mathcal{SAT}_{\beta\pi\mathcal{E}\Rightarrow\times}^* = \mathcal{SAT}_{\beta\pi}^m$ . However, as seen in Ex. 6.3.8, we do not have  $\mathcal{SAT}_{\beta\mathcal{E}\Rightarrow} = \mathcal{SAT}_{\beta}$ .

The important point is that thanks to the stability by union of reducibility candidates for orthogonal constructor rewriting (Thm. 6.3.14), with the instance  $\mathcal{SAT}_{\beta\mathcal{R}\mathcal{E}\Rightarrow\mathcal{C}}$  of  $\mathcal{SAT}_{\mathcal{R}\mathcal{E}}$  we have a reducibility family for a particular case of rewriting which has simple uniform closure properties inspired from those of saturated sets.

## 8 Conclusion and Directions for Future Work

We have presented a notion of non-interaction which allows to define neutral terms and reducibility candidates in a generic way. We have seen that this provides a convenient level of abstraction to prove fundamental properties of reducibility candidates, to compare them with biorthogonals, and to study their stability by union. Moreover, we have proposed a general form of saturated sets based on these notions.

A direct extension of our general framework is to understand how to handle permutative conversions (see [Mat05, Tat07] for recent works on the subject). The first interesting point is to see how these notions of reduction behave w.r.t. to our notions of values. In particular, it seems possible to add them *after* having defined the set of values, which is then preserved. However, we do not know how to interpret this operation. Second, the more important point is to have simple types interpretations for disjunctions. With permutative conversion, this forces [Mat05, Tat07] to define saturated sets at hand, by putting in the closure conditions exactly the terms with good shape. It would be interesting to understand how these closure conditions can be defined more abstractly.

A general line of research is to understand reducibility more abstractly. We see two directions. The first one concerns the connection of reducibility candidates with topological notions. Let us give two examples.

- We associate the set  $F_t =_{\text{def}} \{C \in \mathcal{CR}_{\mathcal{R}\mathcal{E}} \mid t \in C\}$  to each term  $t \in \mathcal{SN}_{\mathcal{R}}$ . Since reducibility candidates are stable by intersection, it is clear that  $F_t$  is a filter in the complete lattice  $(\mathcal{CR}_{\mathcal{R}\mathcal{E}}, \subseteq)$ . It is completely coprime (i.e.  $\mathcal{C} \cup \mathcal{D} \subseteq F_t$  implies  $C \in F_t$  for some  $C \in \mathcal{C}$ ) when  $t$  is a value. Moreover, the stability by union of  $\mathcal{CR}_{\mathcal{R}\mathcal{E}}$  is equivalent to the fact that  $F_t$  is completely coprime for all neutral term  $t \in \mathcal{SN}_{\mathcal{R}}$ .
- In basic topology, the adherence of set  $S$  is the set  $\bar{S}$  of points  $a$  such that  $X \cap S \neq \emptyset$  for all open set  $X$  containing  $a$ . Note that  $S \subseteq \bar{S}$ . Consider the topology  $\Omega$  on  $\mathcal{SN}_{\mathcal{R}}$  generated from the basis of open sets  $(\text{ext}(u))_{u \in \mathcal{V}}$ , where  $\text{ext}(u) =_{\text{def}} \{t \in \mathcal{SN}_{\mathcal{R}} \mid u \in \mathcal{V}(t)\}$ . Hence  $t \in \text{ext}(v)$  for all  $v \in \mathcal{V}(t)$ .

Then, Lem. 4.5.7 implies that reducibility candidates are closed sets on  $\mathcal{SN}_{\mathcal{R}}$  w.r.t. the topology  $\Omega$ , that is  $C = \bar{C}$  for all  $C \in \mathcal{CR}_{\mathcal{R}\mathcal{E}}$ . Indeed, let  $t \in \bar{C}$ . For all  $v \in \mathcal{V}(t)$ , the set  $\text{ext}(v)$  is open by definition, hence there is  $u \in \text{ext}(v) \cap C$ . Since  $v \in \mathcal{V}(u)$ , we get  $v \in C$  by  $(\mathcal{CR}0)$ . It follows that  $\mathcal{V}(t) \subseteq C$ , hence  $t \in C$  by Lem. 4.5.7.

Moreover,  $\mathcal{CR}_{\mathcal{R}\mathcal{E}}$  is the set of all non-empty closed set for the topology  $\Omega$  exactly when it is stable by union. Recall that by Thm. 6.3.4,  $\mathcal{CR}_{\mathcal{R}\mathcal{E}}$  is stable by union exactly when it is the set of all  $C \subseteq \mathcal{SN}_{\mathcal{R}}$  such that  $t \in C$  if and only if  $\mathcal{V}(t) \subseteq C$  for all  $t \in \mathcal{SN}_{\mathcal{R}}$ . Now, let  $C = \bar{C}$  be non empty, and let  $t, u$  such that  $t \lesssim_{\mathcal{V}} u$  and  $u \in C$ . For all  $V \subseteq \mathcal{V}$  such that  $t \in \bigcup_{v \in V} \text{ext}(v)$ , we have  $u \in \bigcup_{v \in V} \text{ext}(v)$ , hence  $(\bigcup_{v \in V} \text{ext}(v)) \cap C \neq \emptyset$  and  $t \in C$ . We deduce that  $C \in \mathcal{CR}_{\mathcal{R}\mathcal{E}}$  by Thm. 6.3.4.

Hence, we have a topological characterization of  $\mathcal{CR}_{\mathcal{R}\mathcal{E}}$  exactly when it is stable by union.



A similar line of research, making connection with the notions of covers and sheaves, has been taken in [Gal95]. These use of topological notions in reducibility deserve to be precisely studied and compared. In particular, it is not clear how to relate the two examples above.

The second line is to understand abstractly the non-interaction property, in connection with the standardization property and rewriting theory. This should help in particular to get a pertinent abstract formulation of saturated sets. The difficulty is to find how to articulate standardization and interactions between terms and contexts in a general way. More precisely, the important point is to understand how standardization can be formulated in a framework specifically based on the duality of values/elimination contexts. A more general view of this duality comes from the notion of polarities. At the computational level, the notion of computational connective [Her08] can be an interesting framework to work on this question. Since the property of stability by union of reducibility candidates makes the connection between standardization and reducibility, it could be interesting to look at sufficient conditions on terms and evaluation contexts such that standardization implies stability by union.

At the logical level, the connections between stability by union and classical logic deserve to be clarified, in particular because stability by union can be used to interpret a form of intuitionistic existential quantification.

Finally, one of the main question on reducibility is to understand how, as an untyped model of typing, it can characterize logical properties of proofs. The difficulty here is that it is difficult, in the framework of the  $\lambda$ -calculus, to characterize logical properties from computational properties of untyped terms. For the fragment of logic without exponentials, Girard's Ludics [Gir01] seems to be a promising approach of this question on which it is appealing to test the ideas developed in this paper.

## References

- [Abe04] A. Abel. Termination Checking with Types. *RAIRO – Theoretical Informatics and Applications*, 38(4):277–319, 2004. Special Issue (FICS'03). [4](#), [10](#)
- [Abe06a] A. Abel. Semi-Continuous Sized Types and Termination. In *Proceedings of CSL'06*, volume 4207 of *LNCS*, pages 72–88. Springer, 2006. [3](#), [39](#)
- [Abe06b] A. Abel. *Type-Based Termination. A Polymorphic Lambda-Calculus with Sized Higher-Order Types*. PhD thesis, LMU University, Munich, 2006. [40](#)
- [AC98] R. M. Amadio and P.-L. Curien. *Domains and Lambda-Calculi*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 1998. [33](#), [35](#)
- [Alt93] T. Altenkirch. *Constructions, Inductive Types and Strong Normalization*. PhD thesis, University of Edinburgh, 1993. [12](#)
- [Bar92] H.P. Barendregt. Lambda Calculi with Types. In S. Abramsky, D.M. Gabbay, and T.S.E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 2. Oxford University Press, 1992. [4](#), [11](#), [12](#)
- [BJO02] F. Blanqui, J.-P. Jouannaud, and M. Okada. Inductive-Data-Types Systems. *Theoretical Computer Science*, 271:41–68, 2002. [17](#), [20](#), [23](#)
- [Bla07] F. Blanqui. Computability Closure: Ten Years Later. In H. Comon, C. Kirchner, and H. Kirchner, editors, *Rewriting, Computation and Proof, Essays Dedicated to*

*Jean-Pierre Jouannaud on the Occasion of his 60th Birthday*, volume 4600 of *LNCS*, pages 68–88. Springer, 2007. [17](#), [20](#)

- [Böh68] C. Böhm. Alcune Proprietà delle Forme  $\beta$ - $\eta$ -Normali nel  $\lambda$ -K-Calcolo. Technical Report 696, Pubblicazioni dell’ Istituto per le Applicazioni del Calcolo, Roma, Italy, 1968. [50](#)
- [Bou85] G. Boudol. Computational Semantics of Term Rewriting Systems. In M. Nivat and J.C. Reynolds, editors, *Algebraic Methods in Semantics*. Cambridge University Press, 1985. [46](#)
- [BR06] F. Blanqui and C. Riba. Combining Typing and Size Constraints for Checking the Termination of Higher-Order Conditional Rewrite Systems. In *Proceedings of LPAR’06*, volume 4246 of *LNAI*, 2006. [3](#), [20](#)
- [CS06] T. Coquand and A. Spiwack. A Proof of Strong Normalisation using Domain Theory. In *Proceedings of LiCS’06*, pages 307–316. IEEE Computer Society, 2006. [6](#)
- [DHR96] V. Danos, H. Herbelin, and L. Regnier. Game Semantics & Abstract Machines. In *Proceedings of LICS’96*, pages 394–405, 1996. [44](#)
- [DK00] V. Danos and J.-L. Krivine. Disjunctive Tautologies as Synchronisation Schemes. In *Proceedings of CSL’00*, volume 1862 of *LNCS*, pages 292–301, 2000. [36](#)
- [Gal89] J.H. Gallier. On Girard’s ”Candidats de Réducibilité”. In P. Odifredi, editor, *Logic and Computer Science*. Academic Press, 1989. [3](#), [4](#), [9](#), [12](#), [20](#), [22](#), [27](#), [28](#), [51](#)
- [Gal91] J.H. Gallier. What’s So Special About Kruskal’s Theorem and the Ordinal  $\Gamma_0$ ? A Survey of Some Results in Proof Theory. *Annals of Pure and Applied Logic*, 53(3):199–260, 1991. [29](#)
- [Gal95] J.H. Gallier. Proving Properties of Typed lambda-Terms Using Realizability, Covers, and Sheaves. *Theoretical Computer Science*, 142(2):299–368, 1995. [3](#), [4](#), [56](#)
- [Gir72] J.-Y. Girard. *Interprétation Fonctionnelle et Élimination des Coupures de l’Arithmétique d’Ordre Supérieur*. PhD thesis, Université Paris 7, 1972. [3](#), [9](#)
- [Gir87] J.-Y. Girard. Linear Logic. *Theoretical Computer Science*, 50:1–102, 1987. [3](#), [36](#)
- [Gir01] J.-Y. Girard. Locus Solum: From the rules of logic to the logic of rules. *Mathematical Structures in Computer Science*, 11(3):301–506, 2001. [56](#)
- [GKK05] J. Glauert, D. Kesner, and Z. Khasidashvili. Expression Reduction Systems and Extensions: An Overview. In *Processes, Terms and Cycles: Steps to the Road of Infinity, Essays Dedicated to Jan Willem Klop, on the Occasion of His 60th Birthday*, volume 3838 of *LNCS*, pages 496–553. Springer, 2005. [46](#)
- [GLT89] J.-Y. Girard, Y. Lafont, and P. Taylor. *Proofs and Types*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 1989. [9](#), [11](#), [20](#), [22](#), [27](#)
- [Her08] H. Herbelin. Duality of Computation and Sequent Calculus: A Few More Remarks. Unpublished manuscript, 2008. [56](#)

- [HL91] G. Huet and J.-J. Lévy. Computations in Orthogonals Systems, I & II. In J. L. Lassez and G. Plotkin, editors, *Computational Logic, Essays in Honour of Alan Robinson*. MIT Press, 1991. [45](#), [46](#)
- [HO93] J. M. E. Hyland and C.-H. Ong. Modified Realizability Toposes and Strong Normalization Proofs. In *TLCA*, LNCS, pages 179–194. Springer, 1993. [3](#)
- [KOvO01] Z. Khasidashvili, M. Ogawa, and V. van Oostrom. Perpetuality and Uniform Normalization in Orthogonal Rewrite Systems. *Information and Computation*, 164(1):118–152, 2001. [4](#), [46](#), [51](#), [54](#)
- [Kri90] J.-L. Krivine. *Lambda-Calcul, Types et Modèles*. Masson, 1990. [4](#), [11](#), [12](#), [50](#)
- [Kri04] J.-L. Krivine. Realizability in Classical Logic. To appear in *Panoramas et synthèses*, Société Mathématique de France, available on HAL, 2004. [3](#), [4](#), [11](#), [12](#), [36](#)
- [LS05] S. Lindley and I. Stark. Reducibility and TT-Lifting for Computation Types. In *TLCA*, volume 3461 of *LNCS*, pages 262–277. Springer, 2005. [4](#)
- [Luo90] Z. Luo. *An Extended Calculus of Constructions*. PhD thesis, University of Edinburgh, 1990. [3](#), [28](#), [44](#), [51](#)
- [Mat98] R. Matthes. *Extensions of System F by Iteration and Primitive Recursion on Monotone Inductive Types*. PhD thesis, Ludwig-Maximilians-Universität München, 1998. [11](#)
- [Mat05] R. Matthes. Non-Strictly Positive Fixed-Points for Classical Natural Deduction. *Annals of Pure and Applied Logic*, 133:205–230, 2005. [4](#), [10](#), [55](#)
- [Mel05] P.-A. Melliès. Axiomatic Rewriting Theory I: A Diagrammatic Standardization Theorem. In *Processes, Terms and Cycles*, volume 3838 of *Lecture Notes in Computer Science*, pages 554–638. Springer, 2005. [4](#), [46](#), [54](#)
- [MV05] P.-A. Melliès and J. Vouillon. Recursive Polymorphic Types and Parametricity in an Operational Framework. In *Proceedings of LiCS’05*, pages 82–91. IEEE Computer Society, 2005. [36](#)
- [Par97] M. Parigot. Proofs of Strong Normalization for Second Order Classical Natural Deduction. *Journal of Symbolic Logic*, 62(4):1461–1479, 1997. [3](#), [36](#)
- [Pit00] A. M. Pitts. Parametric Polymorphism and Operational Equivalence. *Mathematical Structures in Computer Science*, 10:321–359, 2000. [3](#), [36](#)
- [Reg94] L. Regnier. Une équivalence sur les lambda-termes. *Theoretical Computer Science*, 126(2):281–292, 1994. [44](#)
- [Rib07a] C. Riba. *Definitions par Réécriture dans le  $\lambda$ -Calcul: Confluence, Réductibilité et Typage*. PhD thesis, INPL, 2007. [4](#)
- [Rib07b] C. Riba. On the Stability by Union of Reducibility Candidates. In *Proceedings of FoSSaCS’07*, volume 4423 of *LNCS*, 2007. [4](#), [21](#), [32](#), [39](#), [41](#), [42](#), [51](#)
- [Rib07c] C. Riba. Strong Normalization as Safe Interaction. In *Proceedings of LiCS’07*, pages 13–22. IEEE Computer Society, 2007. [3](#), [4](#), [39](#), [41](#), [46](#), [47](#)

- [Rib08] C. Riba. Stability by Union of Reducibility Candidates for Orthogonal Constructor Rewriting. In *Proceedings of CiE'08*, volume 5028 of *LNCS*, pages 498–510. Springer, 2008. Special Session "Higher-type Recursion and Applications". [4](#), [9](#), [21](#), [41](#), [46](#)
- [Tai75] W. W. Tait. A Realizability Interpretation of the Theory of Species. In R. Parikh, editor, *Logic Colloquium*, volume 453 of *LNCS*, pages 240–251, 1975. [3](#), [9](#), [12](#)
- [Tat07] M. Tatsuta. Simple Saturated Sets for Disjunction and Second-Order Existential Quantification. In *Proceedings of TLCA'07*, volume 4583 of *LNCS*, pages 366–380. Springer, 2007. [3](#), [39](#), [40](#), [44](#), [55](#)
- [Ter03] Terese. *Term Rewriting Systems*. Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2003. M. Bezem, J.W. Klop and R.C. de Vrijer, eds. [4](#), [6](#)
- [VM04] J. Vouillon and P.-A. Melliès. Semantic Types: A Fresh Look at the Ideal Model for Types. In *Proceedings of POPL'04*. ACM, 2004. [3](#), [4](#), [8](#), [36](#), [47](#)
- [Vou04] J. Vouillon. Subtyping Union Types. In *Proceedings of CSL'04*, volume 3210 of *LNCS*, pages 415–429. Springer Verlag, 2004. [3](#), [4](#), [8](#), [36](#), [47](#)
- [vRS95] F. von Raamsdonk and P. Severi. On Normalisation. Technical Report CS-R9545, CWI, 1995. [4](#), [24](#)
- [Wer94] B. Werner. *Une Théorie des Contructions Inductives*. PhD thesis, Université Paris 7, 1994. [3](#), [19](#), [40](#)